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ASYMPTOTIC ANALYSIS OF THREE-DIMENSIONAL WATER IMPACT

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Introduction

The water impact phenomenon is still a challenge for researchers. The useful model of impact was introduced by Wagner (1932). The Wagner model is based on the so called flat-disc approximation, where the real shape of the impacting body is substituted in the contact region by a flat disc and the boundary conditions are linearized and imposed on the initially undisturbed free surface. The disc shape is unknown in advance and has to be determined from an additional condition. The Wagner theory is formally valid during the initial stage of blunt-body impact. This is a basis for development of more adequate models. Moreover, Wagner theory itself provides quite reasonable estimates of hydrodynamic loads on entering body, which are helpful at the design stage.

Both two-dimensional and axisymmetric impact problems have been well studied within Wagner theory. The gained experience made it possible to generalize the basic theory taking into account real shape of the entering body and non-linear effects. For the last three years, the three-dimensional Wagner problem has been receiving an increasing attention. Several exact solutions of this problem were presented in Scolan and Korobkin (2000, 2001a) and the energy distribution between the 3D jets and the bulk of the fluid was analyzed by Scolan and Korobkin (2001b). These results were obtained by an inverse method. They are limited to the case of elliptic contact lines between the liquid free surface and the entering body. However, this restriction does not keep us from studying numerous shapes of practical interest.

If the shape of the contact line is not given, we face the direct Wagner problem. The method of its solution is far to be straightforward: we are still looking for efficient algorithms. The three-dimensional Wagner problem is nonlinear and still complicated for analysis. In this paper we suggest to linearize this problem on the basis of a known solution and to start with the study of the linearized problem. The linearization is performed about an axisymmetric solution. The linearized problem formally describes the entry of an almost axisymmetric body into the liquid. The solution of the linearized Wagner problem is obtained and compared to exact solutions available. The second order solution is also considered, in order to obtain the leading order correction to the hydrodynamic force on the body. The method developed can be helpful to test other numerical approaches and to optimize the shape of the entering body utilizing three-dimensional effects.

New formulation of the Wagner problem

The three-dimensional problem of unsteady liquid flow arising when a blunt body enters an ideal incompressible liquid through the free surface, is considered. Initially the liquid is at rest and occupies the lower half-space, $z < 0$. Position of the entering body is given by the equation $z = f(x, y) - h(t)$, where the function $f(x, y)$ describes the body shape and $h(t)$ is the penetration depth of the body, $h(0) = 0$. External mass forces and surface tension are neglected. The liquid flow caused by the impact is assumed irrotational.

Within the Wagner approach the flow is described by the displacement potential $\varphi(x, y, z, t)$, for which the boundary-value problem has the form (see Korobkin, 1982 and Howison *et al.*, 1991 for details)

$$\begin{cases} \varphi_{,xx} + \varphi_{,yy} + \varphi_{,zz} = 0 & z < 0 \\ \varphi = 0 & z = 0,\ (x, y) \in \mathrm{FS}(t) \\ \varphi_{,z} = f(x, y) - h(t) & z = 0,\ (x, y) \in \mathrm{D}(t) \\ \varphi \to 0 & (x^2 + y^2 + z^2) \to \infty \end{cases} \tag{1}$$

The elevation of the disturbed free surface $Z(x, y, t)$ is obtained from the linearized kinematic boundary condition $Z = \varphi_{,z}(x, y, 0, t)$, where $(x, y) \in$FS(t), and the pressure distribution $p(x, y, z, t)$ from the linearized Bernoulli equation $p = -\gamma\varphi_{,tt}$, where γ is the liquid density. The vertical component of the hydrodynamic force $F(t)$ acting on the entering body is evaluated by integration of the pressure $p(x, y, 0, t)$ over the contact region D(t).

The division of the liquid boundary into a free surface FS(t) and a contact region D(t) is unknown in advance and must be determined together with the displacement potential. The classical Wagner condition,

$Z(x,y,t)=f(x,y)-h(t)$ along the contact line $\Gamma(t)=\partial\mathrm{D}(t)$, is reduced to the condition that the displacement potential is continuously differentiable up to the liquid boundary, $z\le 0$.

The boundary-value problem (1) leads to the integral equation

$$\frac{1}{2\pi}\int\int_{D(t)}\frac{s(x_0,y_0,t)\mathrm{d}x_0\mathrm{d}y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=h(t)-f(x,y)\qquad((x,y)\in\mathrm{D}(t)),\tag{2}$$

where $s(x,y,t)=\Delta_2\varphi=\varphi_{,xx}(x,y,0,t)+\varphi_{,yy}(x,y,0,t)$. The function $s(x,y,t)$ is square integrable. Once the integral equation (2) has been solved, the displacement potential $\tilde{\varphi}(x,y,t)=\varphi(x,y,0,t)$ in the contact region $\mathrm{D}(t)$ is calculated from the solution of the following boundary-value problem for the Poisson equation

$$\begin{cases}\Delta_2\tilde{\varphi}=s(x,y,t) & (x,y)\in\mathrm{D}(t)\\ \tilde{\varphi}=0 & (x,y)\in\Gamma(t)\\ \dfrac{\partial\tilde{\varphi}}{\partial n}=0 & (x,y)\in\Gamma(t)\end{cases}\tag{3}$$

The latter condition follows from the continuity of the displacements at the contact line. This equation is used to determine the position of the contact line $\Gamma(t)$.

We are concerned with the case of an almost axisymmetric entering body, shape of which is described by equation

$$f(x,y)=f_0(r)+\epsilon F(r,\theta),\tag{4}$$

where $x=r\cos\theta$, $y=r\sin\theta$, $\epsilon\ll 1$, the functions $f_0(r)$ and $F(r,\theta)$ are smooth and $F(r,\theta)=O(1)$, $f_0(r)=O(1)$, $F(0,\theta)=0$, $f_0(0)=0$, $f_0'(r)=df_0/dr\ll 1$ in a small vicinity of the impact point. The positive non-dimensional parameter ϵ is considered here as the parameter of linearization. It is important to notice that the region $\mathrm{D}(t)$, where the functions $s(x,y,t)$ and $\tilde{\varphi}(x,y,t)$ must be determined, is also dependent on the small parameter ϵ. Therefore, asymptotic methods cannot be applied directly to equations (2)-(3).

In order to avoid this difficulty, the parametrization of the contact region $\mathrm{D}(t)$ is introduced. The conformal mapping $\zeta=g(\omega,t)$ is used, where $\zeta=x+iy$, $\omega=\xi+i\eta$, $|\omega|\le 1$, $g(0,t)=0$ and $g(\omega,t)=g_1(\xi,\eta,t)+ig_2(\xi,\eta,t)$. The function $g(\omega,t)$ maps the unit circular disc $|\omega|<1$ onto the contact region $\mathrm{D}(t)$. This function must be determined together with the displacement potential. In the new coordinate system (ξ,η) equations (2)-(3) take the forms

$$\frac{1}{2\pi}\int\int_{|\omega_0|<1}\frac{S(\xi_0,\eta_0,t)\mathrm{d}\xi_0\mathrm{d}\eta_0}{|g(\omega,t)-g(\omega_0,t)|}=Y(\xi,\eta,t)\qquad(\xi^2+\eta^2<1),\tag{5}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\alpha^2}=S(\rho,\alpha,t)\quad(\rho<1),\qquad\Phi=\frac{\partial\Phi}{\partial\rho}=0\quad(\rho=1),\tag{6}$$

where $S(\xi,\eta,t)=s[g_1(\xi,\eta,t),g_2(\xi,\eta,t),t]$, $Y(\xi,\eta,t)=h(t)-f[g_1(\xi,\eta,t),g_2(\xi,\eta,t)]$, $\Phi(\rho,\alpha,t)=\tilde{\varphi}[g_1(\xi,\eta,t),g_2(\xi,\eta,t),t]$ and $\xi=\rho\cos\alpha$, $\eta=\rho\sin\alpha$.

The Wagner problem is formulated now as follows: determine the function $\Phi(\rho,\alpha,t)$ and the conformal mapping $g(\omega,t)$, which satisfy equations (5)-(6) in the unit circle. Within this formulation the Wagner problem is suitable for an asymptotic analysis.

Asymptotic analysis

An approximate solution of equations (5)-(6) as $\epsilon\to 0$ is sought in the form

$$\begin{cases}\Phi(\rho,\alpha,t)=\Phi_0(\rho,t)+\epsilon\Phi_1(\rho,\alpha,t)+\epsilon^2\Phi_2(\rho,\alpha,t)+O(\epsilon^2),\\ S(\rho,\alpha,t)=S_0(\rho,t)+\epsilon S_1(\rho,\alpha,t)+\epsilon^2 S_2(\rho,\alpha,t)+O(\epsilon^2),\\ g(\omega,t)=a(t)[\omega+\epsilon H(\omega,t,\epsilon)],\\ H(\omega,t,\epsilon)=H_0(\omega,t)+\epsilon H_1(\omega,t)+O(\epsilon^2)\end{cases}\tag{7}$$

with the leading order terms reflecting the fact that the solution is axisymmetric for $\epsilon=0$.

Substituting the asymptotic expansions (7) into equations (5)-(6) and collecting the terms of the same order as ϵ^j, $j\ge 0$, the successive boundary-value problems are derived for the unknown functions $\Phi_j(\rho,\alpha,t)$, $S_j(\rho,\alpha,t)$ and $H_j(\omega,t)$. We obtain

$$\frac{1}{2\pi}\iint_{|\omega_0|<1}[S_j(\xi_0,\eta_0,t)-S_0(\xi_0,\eta_0,t)\Re(T_{j-1})]\frac{\mathrm{d}\xi_0\mathrm{d}\eta_0}{|\omega-\omega_0|}=-\frac{a^2}{\rho}f_0'(a\rho)\Re(\bar{\omega}H_{j-1})-aK_j(\rho,\alpha,t), \tag{8}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_j}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi_j}{\partial\alpha^2}=S_j(\rho,\alpha,t) \qquad (\rho<1), \tag{9}$$

$$\Phi_j=\frac{\partial\Phi_j}{\partial\rho}=0 \qquad (\rho=1). \tag{10}$$

Here $T_j(\omega,\omega_0,t)=[H_j(\omega,t)-H_j(\omega_0,t)]/(\omega-\omega_0)$, where $H_{-1}\equiv 0$. The functions $K_j(\rho,\alpha,t)$ are dependent on previous approximations: $H_n(\omega,t)$, $0\le n\le j-2$ and $S_m(\rho,\alpha,t)$, $0\le m\le j-1$, and are considered as given. In particular, $K_0(\rho,\alpha,t)=f_0(a\rho)-h(t)$ and $K_1(\rho,\alpha,t)=F(a\rho,\alpha)$.

The solutions of the boundary-value problems (8) - (10) are sought in the form

$$\Phi_j(\rho,\alpha,t)=\Re\sum_{n=0}^{\infty}\Phi_n^{(j)}(\rho,t)e^{in\alpha}, \quad H_j(\omega,t)=\sum_{n=1}^{\infty}a_n^{(j)}(t)\omega^n, \quad \text{where } F(r,\theta)=\Re\sum_{n=1}^{\infty}A_n(r)e^{in\theta}.$$

The analytical solutions are obtained at zeroth ($j=0$) and first ($j=1$) orders. In particular, the function $a(t)$ and the coefficients of the conformal mapping $a_n^{(1)}(t)$ are given as

$$\int_0^{\frac{\pi}{2}}\sin\beta f_0[a(t)\sin\beta]\mathrm{d}\beta=h(t), \qquad a_n^{(1)}(t)=-\frac{\dot{a}}{a\dot{h}}\int_0^{\frac{\pi}{2}}(\sin\beta)^n A_{n-1}[a\sin\beta]\mathrm{d}\beta \qquad (n\ge 1). \tag{11}$$

It should be noted that the axisymmetric Wagner problem solution is reduced to a simple algebraic equation for $a(t)$. The equations for $a_n^{(1)}(t)$ indicate that axisymmetric solutions of the Wagner problems are unstable with respect to small imperfections of the body shape if $\dot{a}(t)\gg 1$.

The hydrodynamic force $F(t)$ is obtained as $F(t)=F_a(t)+\epsilon F_1(t)+\epsilon^2F_2(t)+O(\epsilon^3)$ where $F_a(t)$ is the force on axisymmetric body ($\epsilon=0$). It is shown that $F_1(t)\equiv 0$. The second order contribution $F_2(t)$ was calculated for elliptic paraboloid entry problem.

Elliptic paraboloid impact

The shape function $f(x,y)$ of the elliptic paraboloid is decomposed according to equation (4) as follows

$$f(x,y)=\frac{x^2}{2r_x}+\frac{y^2}{2r_y}=\frac{r^2}{2R}+\epsilon\frac{r^2}{2R}\cos 2\theta,$$

where r_x and r_y are the curvature radii of the body at the impact point, $r_y\ge r_x$, $\epsilon=(r_y-r_x)/(r_y+r_x)$ and $R=2r_yr_x/(r_y+r_x)$. The zeroth order solution gives

$$a(t)=\sqrt{3Rh(t)}, \qquad F_a(t)=\frac{8\gamma}{45R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[a^5(t)\right],$$

which corresponds to well-known axisymmetric solution. The first order solution gives no contribution to the hydrodynamic force and provides the shape of the contact line $\Gamma(t)$ in the polar coordinates

$$r=a(t)\left[1-\frac{2}{5}\epsilon\cos(2\theta)+O(\epsilon^2)\right],$$

which coincides with the corresponding asymptotics obtained from the exact solution by Scolan & Korobkin (2001a). The second order solution gives no contribution to the area of the contact region D(t) but provides the correction to the hydrodynamic force

$$F_2(t)=\frac{8\gamma}{75R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a^5(t)],$$

which is in agreement with the exact formula $F(t)=(\gamma G(\epsilon)/R)[a^5(t)]_{,tt}$ following from the analysis by Scolan & Korobkin (2001a). The function $G(\epsilon)$ (dotted line) together with its zeroth (thin solid line) and second (thick solid line) approximations is shown in figure 1. It is seen that the second order approximation of the hydrodynamic force can be used with relative error less than 5% up to $\epsilon=0.56$.

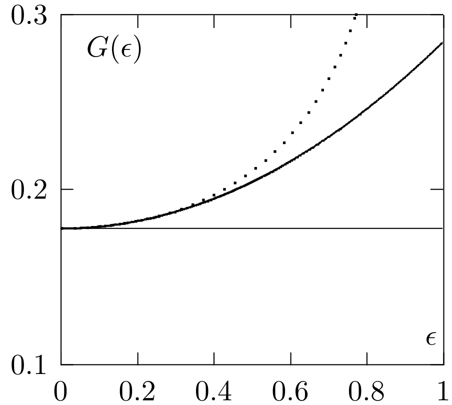

Figure 1.

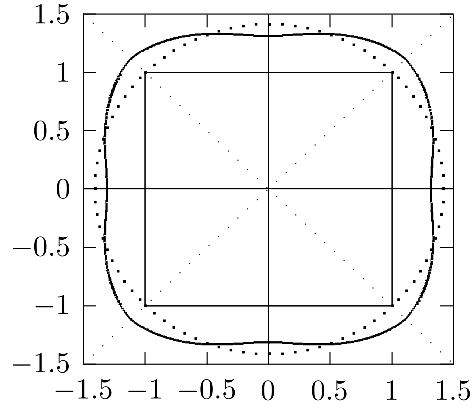

Figure 2.

Water entry of pyramid

We consider a pyramid with square cross section

$$z = rw(\theta)\tan\Theta - h(t), \quad w(\theta) = \frac{\sqrt{8}}{\pi}\left(1 + \frac{2}{15}\sum_{m=1}^{\infty}\frac{15}{16m^2-1}(-1)^{m+1}\cos 4m\theta\right),$$

where Θ is the deadrise angle. The small parameter is formally introduces as $\epsilon = 2/15$.

The pyramid entry problem is self-similar. It is convenient to introduce new variables as $r_1 = rh^{-1}(t)\tan\Theta$, $z_1 = zh^{-1}(t)\tan\Theta$ and $\varphi_1(r_1,\theta,z_1) = \varphi h^{-2}(t)\tan\Theta$. In the stretched coordinates the contact line is only dependent on the formal parameter ϵ. The first order solution provides the contact line in the parametric form

$$x_1 + iy_1 = \sqrt{2}e^{i\alpha} + \sum_{m=1}^{\infty} b_m e^{i[4m+1]\alpha}, \quad b_{m+1} = -b_m\frac{16m^2-1}{8(2m+3)(m+1)}, \quad b_1 = -\frac{\sqrt{2}}{12}.$$

In figure 2 the contact line is shown with thick solid curve, the zeroth order approximation provides the circle (dotted line), the square corresponds to the intersection line between the pyramid and the undisturbed free surface. The approximate contact line is smooth although the intersection line has corner points. Experiments with transparent pyramids are highly required to justify the obtained results.

The hydrodynamic force on a pyramid with square cross section is given as

$$F(t) = \gamma N \tan^{-3}\Theta \frac{\mathrm{d}^2}{\mathrm{d}t^2}[h^4(t)],$$

where N is an universal constant which is suggested to recover using experimental data.

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