Uniqueness in linearized problems of a floating body supported by an air cushion

Nikolay KUZNETSOV

Laboratory for Mathematical Modelling of Wave Phenomena, Institute of Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol'shoy pr. 61, St Petersburg 199178, RF;
E-mail: nikuz@wave.ipme.ru *Fax: 7 (812) 3214771*

Summary. The question of uniqueness is considered for linearized problems describing the radiation and scattering of time-harmonic water waves by a floating body supported by an air cushion. It is shown that the method developed by John for proving the uniqueness theorem is applicable to the coupled boundary value problem presented by Newman at the 15th Workshop on WWFB.

1. Introduction. In studies of water waves interacting with obstacles the question of uniqueness in linearized problems is not yet fully answered despite its importance (see Ursell [6], where this question is placed first in the list of unfinished problems). The difficulty of this question may be illustrated by the fact that during the fifty years between 1950, when the pioneering papers by John and Ursell appeared, and 2000 less than twenty works were published on this topic (see the book by Kuznetsov, Maz'ya, and Vainberg [2]). Besides, new problems arise and require to give an answer to the same question. Thus several talks at the WWFB Workshops 2000 and 2001 were concerned with a couple of mathematical models, describing the radiation and scattering of water waves by a floating body supported by an air cushion.

Here we consider the question of uniqueness for the three-dimensional boundary value problem, analogous to the two-dimensional one proposed by Newman [4]. It provides the coupling between the acoustic and water waves, of which the former are confined to an air chamber A beneath a floating body D, and the latter propagate in an open sea W of a constant depth d. We assume that D is rigid and is bounded by a smooth surface S, which is the union of three disjoint parts: (1) S_a, separating D from the atmosphere; (2) S_w, separating D from W; (3) S_c, separating D from A (see Fig. 1). In its turn, the water domain W is bounded by the mean free surface F, coinciding with the (x_1, x_2)-plane outside the floating body (the origin of Cartesian coordinates (x_1, x_2, y) is placed in F, and the y-axis is directed upwards), by rigid surfaces $S_w \cup \{y = -d\}$ (the latter is the bottom), and by an interface I described below. Finally, A is bounded from above by S_c, and from below by the horizontal air-chamber/water interface I, whose level is above or below (as shown in Fig. 1) the sea level, depending on whether the air pressure within A is smaller or greater, respectively, than the atmospheric

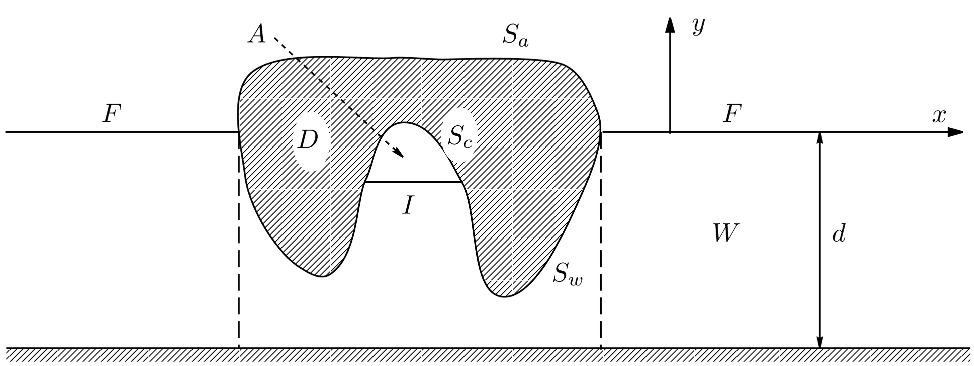

Figure 1: A definition sketch of the geometry and the corresponding notation

pressure. If the pressure in A is equal to that in the atmosphere, then the interface level coincides with the sea level.

The assumptions usual in the linear acoustics and in the linear water-wave theory are adopted here. With the time-dependence represented by the factor $\exp\{-\mathrm{i}\omega t\}$, the velocity fields in A and W are given as the gradients of the complex-valued potentials $\Phi(x, y)$ and $\phi(x, y)$, respectively, where $x = (x_1, x_2)$. The governing equations are as follows:

$$\nabla^2 \Phi + k_c^2 \Phi = 0 \;\text{ in }\; A, \quad \nabla^2 \phi = 0 \;\text{ in }\; W, \tag{1}$$

where $k_c = \omega/c$ is the acoustic wavenumber and c is the sound velocity. The no-flow conditions,

$$\partial\Phi/\partial n = 0 \;\text{ on }\; S_c, \quad \partial\phi/\partial n = 0 \;\text{ on }\; S_w \cup \{y = -d\}, \tag{2}$$

are natural on the rigid surfaces because we are interested in proving the uniqueness theorem for the radiation and scattering problems. The linearized kinematic and dynamic conditions combine in the usual way on the free surface to give

$$\phi_y - \omega^2 \phi = 0 \;\text{ on }\; F, \tag{3}$$

where the units are chosen so that the acceleration due to gravity is equal to one. Two conditions, coupling the velocity fields in the chamber and in the water domain, are as follows:

$$\Phi_y = \phi_y, \quad \rho_c \omega^2 \Phi = \rho_w(\omega^2 \phi - \phi_y) \quad \text{on} \;\; I. \tag{4}$$

By ρ_c and ρ_w we denote the densities of air in the chamber and of water, respectively; the aerostatic pressure in A is neglected when deriving the last condition from the kinematic and dynamic ones. In order to complete the problem's statement the radiation condition for water waves must be imposed, and we use the form introduced by John [1]:

$$\int_{C_\sigma} \left|\phi_{|x|} - \mathrm{i}k_w \phi\right|^2 \mathrm{d}S = o(1) \quad \text{as} \;\; \sigma \to \infty. \tag{5}$$

Here $C_\sigma = \{x_1^2 + x_2^2 = \sigma^2, \; -d < y < 0\}$, and k_w is the only positive root of $k_w \tanh k_w d = \omega^2$.

2. The uniqueness theorem. The aim of this note is to demonstrate that the method developed by John [1] for the water-wave problem, is also applicable to problem (1)–(5). This method relies essentially on the following geometric assumptions usually referred to as *John's conditions*: (*i*) any straight segment parallel to the y-axis, and connecting F with the bottom $\{y = -d\}$, lies within W except for its ends; (*ii*) F is a connected subset of the (x_1, x_2)-plane. These assumptions allow to define the following function

$$u(x) = \int_{-d}^{0} \phi(x, y) \cosh k_w(y + d)\, \mathrm{d}y$$

throughout F, and to show that *u is identically equal to zero if the flux of energy to infinity vanishes*; that is,

$$\operatorname{Im} \int_{C_\sigma} \phi_{|x|}\, \overline{\phi}\, \mathrm{d}S = 0 \text{ for sufficiently large } \sigma > 0. \tag{6}$$

From the fact that $u = 0$ on F, John [1] derived an inequality between the potential energy and the part of kinetic energy enclosed in the portion of W, lying strictly below F. This inequality, together with (6), leads to a contradiction unless $\phi = 0$ in W. It is important that all considerations in [1], except for the derivation of (6), use the Laplace equation, the free surface and the bottom boundary conditions, and the radiation condition only in the domain, lying outside the vertical cylinder through the body's water-line (it is shown by dashed lines in Fig. 1). Therefore, these considerations may be also applied to problem (1)–(5). Hence, for establishing the uniqueness theorem for this problem, it remains only to prove the following lemma.

If the (x_1, x_2)-plane is not tangent to S at any point of the water-line, and the same is true for the plane, to which I belongs, then (6) *holds for a solution to problem* (1)–(5).

Proof. It follows from considerations in [2], Chapter 3, that the assumption made about the intersections of S with the (x_1, x_2)-plane and with the plane, to which I belongs, allows to apply the second Green's identity to ϕ and $\overline{\phi}$ in the truncated water domain $W \cap \{|x| < \sigma\}$. Here σ is chosen so large that C_{σ_*} does not intersect D for all $\sigma_* \geq \sigma$. After application of the boundary conditions (2) and (3) for ϕ, Green's identity takes the following form

$$-\mathrm{Im} \int_{C_\sigma} \phi_{|x|} \, \overline{\phi} \, \mathrm{d}S = \mathrm{Im} \int_I \phi_y \, \overline{\phi} \, \mathrm{d}x.$$

By using (4), we get that the last expression is equal to

$$\mathrm{Im} \int_I \Phi_y \, \left[(\rho_c/\rho_w) \, \overline{\Phi} + \omega^{-2} \, \overline{\Phi}_y \right] \, \mathrm{d}x = (\rho_c/\rho_w) \, \mathrm{Im} \int_I \Phi_y \, \overline{\Phi} \, \mathrm{d}x.$$

Then the second Green's identity, applied to Φ and $\overline{\Phi}$ in A, transforms this into

$$(\rho_c/\rho_w) \, \mathrm{Im} \int_{S_c} (\partial \Phi / \partial n) \, \overline{\Phi} \, \mathrm{d}S = 0,$$

where the equality is a consequence of (2) for Φ. The proof is complete.

The just proven lemma, and the John's considerations, outlined at the beginning of this section, allow to formulate the following uniqueness theorem.

Let (i)*,* (ii)*, and the geometric assumptions of lemma hold. Then problem* (1)–(5) *has only a trivial solution.*

3. Discussion. For the two-dimensional and axisymmetric water-wave problems, it is demonstrated in Simon and Ursell [5], and in Kuznetsov and Simon [3], respectively, that lines inclined to the vertical can be used instead of the vertical ones, thus enlarging the set of the body's geometries for which the uniqueness theorem holds. This can also be done for the two-dimensional and axisymmetric versions of problem (1)–(5). In particular, the uniqueness theorem is true for the simple geometry considered by Newman [4].

Less obvious generalisation of the uniqueness theorem concerns the problem of the coupled air/water motion in the presence an air-chambered body, *freely floating* in water. Such a theorem was proved by John [1] for a freely floating rigid body, having no air chamber. Another possible generalisation is to include the aerostatic pressure into the second condition (4) on the interface I. More complete results in these directions will be presented at the Workshop.

References

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Discussion Sheet

Abstract Title :	Uniqueness in linearized problems of a floating body supported by an air cushion		
(Or) Proceedings Paper No. :	24	***Page :***	091
First Author :	Kuznetsov, N.		
Discusser :	Maureen McIver		

Questions / Comments :

If you take limit $\rho c/\rho w \rightarrow 0$, the problem for Φ decouples and becomes that for water waves with 2 bodies in the free surface. In this case it has only been proved that there are bands of frequencies of uniqueness. It is surprising that such a strong result exists for the more complicated problem. Can you comment on this?

Author's Reply :
(If Available)

The speed of sound tends to infinity as $\rho_c \rightarrow 0$, and so $\kappa_c \rightarrow 0$. Therefore, one obtains the Laplace equation in the air chamber is the limit. The limit form of the coupling conditions provides that Φ is the analytic continuation of ϕ into the air chamber. Then the original John's theorem can be applied, thus guaranteeing the uniqueness.

Questions from the floor included; Nick Newman.