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# MIXING LAYER AT FREE SURFACE

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**SUMMARY**

A simple mathematical model for the evolution of a mixing layer with pressure gradient is developed. The problem of mixing layer interaction with a free surface in water and its transition in a turbulent surface jet is considered; in particular, the velocity field and basic characteristics of turbulent flow in two-dimensional mixing layers and surface jets are obtained.

## 1. INTRODUCTION

The interaction of turbulent flows with a free boundary is an important part of realistic wave description in many applications. The turbulence in a surface layer may be generated in many ways (moving vessels, topography, breaking waves etc.). This problem was studied theoretically and experimentally in [1-6]. Recent experimental methods such as Particle Image Velocimetry (PIV) reveal the very complicated flow patterns of turbulent flow in the surface layer in spilling breakers, hydraulic jumps, turbulent bores [7]. It is clear that the main features of the turbulent layer evolution can be simulated only by the rather complicated mathematical models.

The goal of the present paper is to develop the mathematical model of turbulent layer evolution in open channel flow, which is suitable for the description of velocity fields and turbulence characteristics in mixing layers and surface jets, but it should be simple enough to find some solutions in explicit form. For this purpose we consider the two–level flow description. As the first step, the three–layer shallow water equations, which describe the mean flow evolution with entrainment of fluid from the potential layers into the turbulent interlayer, are derived. Then, the mean turbulent energy and the thickness of the turbulent layer, which have been found from the layered model of flow, are used as the scales in a two-dimensional turbulent flow model.

For the three–layer shallow water approximation, the approach [8], developed for stratified two–layer flows of miscible fluid, is used. The main idea here is in using the total conservation laws of mass, momentum and energy to find the mean flow in the turbulent intermediate layer. The boundary layer approximation for the Reynolds equations is based on the hypothesis from [9] about Reynolds stresses expression in free shear turbulent flows. For steady–state flows, the problem on velocity field restoring in the turbulent layer is reduced to the semilinear initial–boundary problem for a hyperbolic system of differential equations. This approach is applied to the problem on the mixing layer evolution at a free surface and its transition into a surface jet.

## 2. SHALLOW WATER EQUATIONS

We consider the problem of the formation of a mixing layer in an incompressible homogeneous fluid for open channel flows. Aeration of the flow is ignored, and, hence, the liquid density $\rho$ is constant ($\rho \equiv 1$). We apply the three-layer scheme of flow, in which the mixing layer is considered as an intermediate turbulent layer between two layers with potential flow. For upper and lower layers we use the nonhomogeneous shallow water equations in which the entrainment of fluid from the layers into the mixing layer is taken into account. To describe the evolution of the averaged quantities in the mixing layer, we add to the system the total conservation laws of mass, momentum and energy [10]. Under the assumption of hydrostatic pressure distribution in the layers, the governing equations are written as

$$
\begin{aligned}
&h_t^+ + (h^+u^+)_x = -\chi^+\\
&h_t^- + (h^-u^-)_x = -\chi^-\\
&u_t^+ + \left(0.5{u^+}^2 + gH\right)_x = 0\\
&u_t^- + \left(0.5{u^-}^2 + gH\right)_x = 0\\
&H_t + \bar{Q}_x = 0\\
&\bar{Q}_t + \left(h^+{u^+}^2 + \eta\bar{u}^2 + h^-{u^-}^2 + 0.5gH^2\right)_x = 0\\
&(h^+{u^+}^2 + \eta(\bar{u}^2 + q^2) + h^-{u^-}^2 + gH^2)_t +\\
&\quad + (h^+{u^+}^3 + \eta\bar{u}(\bar{u}^2 + q^2) + h^-{u^-}^3 + 2gH\bar{Q})_x = -\bar{\varepsilon}
\end{aligned}
\qquad (1)
$$

Here $t$ is the time, $x$ is the horizontal coordinate, $g$ is the gravity acceleration, $h^+$, $h^-$ are the depths, $u^+$, $u^-$ are the mean horizontal velocities in the up-

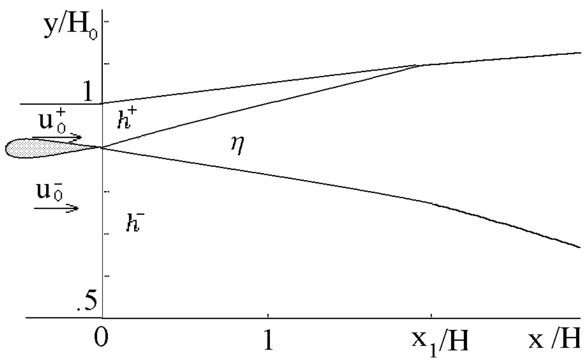


Figure1: Mixing layer at a free surface
$(u_0^+/\sqrt{gH_0} = 2,\ u_0^-/\sqrt{gH_0} = 0.7,\ h_0^-/H_0 = 0.9.)$

per and lower layer, respectively, $\eta$ is the thickness and $\bar{u}$ is the mean horizontal velocity in the interlayer, $H = h^+ + \eta + h^-$ is the total depth and $Q = h^+u^+ + \eta\bar{u} + h^-u^-$.
The entrainment rate $\chi^\pm$ is supposed to be proportional to the root-mean-square velocity $\bar{q}$ of turbulent flow:

$$\chi^\pm = \sigma_0 \bar{q},$$

where the coefficient $\sigma_0 = 0.15$ characterizes the ratio of the vertical and horizontal scales in the shallow water approximation, and it may be eliminated from (1) by replacing of independent variables. The energy dissipation term $\bar{\varepsilon}$ is taken in the form:

$$\bar{\varepsilon} = \theta \bar{q}^3, \quad \theta \equiv \text{const}$$

Note that (1) describes the mixing layer evolution for $h^\pm > 0$. If, say, $h^+ = 0$ at $x = x_1$, we have the transition of the mixing layer in a surface jet and (1) will be correct for the surface jet with $h^+ \equiv 0,\ \chi^+ \equiv 0$ (Fig 1).

## 3. MIXING LAYER

### 3.1. MEAN FLOW EVOLUTION

A steady-state two-dimensional mixing layer forms when two layers of fluid with depths $h_0^+,\ h_0^-$ and velocities $u_0^+,\ u_0^-$ merge at $x = 0$ (Fig 1). As a consequence of (1) we have the following relations for mean characteristics of the flow

$$\begin{aligned} & h^+u^+ + 0.5Q = h_0^+u_0^+ = Q^+ \\ & h^-u^- + 0.5Q = h_0^-u_0^- = Q^- \\ & 0.5{u^+}^2 + gH = 0.5{u_0^+}^2 + gH_0 = J^+ \\ & 0.5{u^-}^2 + gH = 0.5{u_0^-}^2 + gH_0 = J^- \\ & h^+{u^+}^2 + \eta\bar{u}^2 + h^-{u^-}^2 + 0.5gH^2 = \\ & h_0^+{u_0^+}^2 + h_0^-{u_0^-}^2 + 0.5gH_0^2 = F \end{aligned} \qquad (2)$$

Here $Q = \eta\bar{u}$, $H_0 = h_0^+ + h_0^-$ and all unknown variables can be expressed from (2) as functions of $Q$. For the stationary mixing layer (1) is reduced to the system of ODE

$$\begin{aligned} \frac{dQ}{dx} &= 2\sigma_0\bar{q} \\ \frac{d\bar{q}}{dx} &= \frac{\sigma_0}{Q}\Big(f_l(Q) - (1+\delta)\sigma_0\bar{q}^2\Big), \end{aligned} \qquad (3)$$

where $\delta = \theta/(2\sigma_0)$, $f_l(Q) = \bar{u}^2 + 0.5{u^+}^2 + 0.5{u^-}^2 - \bar{u}(u^+ + u^-)$. Eqs. 3 may be rewritten as a linear ODE with the unknown function $\bar{q}^2 = \bar{q}^2(Q)$

$$Q\frac{d\bar{q}^2}{dQ} = \Big(f_l(Q) - (1+\delta)\sigma_0\bar{q}^2\Big), \qquad (4)$$

which has the bounded solution for $Q > 0$

$$\bar{q}^2(Q) = Q^{-(1+\delta)}\int_0^Q s^\delta f_l(s)\, ds.$$

Note that $f_l(0) = 0.25(u_0^+ - u_0^-)^2$, $\bar{q}^2(0) = f_l(0)/(1+\delta)$ and the entrainment in the mixing layer starts at $x = 0$ with the finite rate

$$\bar{q}_0 = (u_0^+ - u_0^-)/(1+\delta)^{\frac{1}{2}}.$$

The dependence of the mean flow characteristics on $x$ can be restored from the quadrature formula

$$x = \int_0^Q \frac{ds}{2\sigma_0\bar{q}(s)}$$

The behaviour of the free surface $H = H(x)$ depends on the sign of the determinant

$$\Delta_l = 1 - \frac{g\eta}{\bar{u}^2} - \frac{gh^+}{{u_0^+}^2} - \frac{gh^-}{{u_0^-}^2}.$$

In subcritical flows ($\Delta_l > 0$) the total depth $H(x)$ increases and in supercritical flows ($\Delta_l < 0$) the function $H(x)$ decreases. The dependence of the total depth and the boundaries of the mixing layer is shown in Fig.1. The upper boundary of the mixing layer reaches the free boundary at $x = x_1$.

### 3.2. VELOCITY FIELD IN MIXING LAYER

The distribution of the mean quantities in the mixing layer have been found above. We use the boundary layer approximation to calculate the horizontal and vertical velocity components $u = u(x,y)$, $v = v(x,y)$ as well as the root-mean-square velocity $q = q(x,y)$ in the free turbulent flow [9]. For steady-state flows the governing equations take the form ($\rho \equiv 1$):

$$\begin{aligned} & u_x + v_y = 0 \\ & uu_x + vu_y + \tau_y = -p_x^* \\ & uq_x + vq_y + \tau u_y = -\varepsilon, \end{aligned} \qquad (5)$$

where the Reynolds stress $\tau$ is expressed by the formula

$$\tau = -\sigma \bar{q} q, \quad \sigma = \sigma_0 \mathrm{sgn} u_y . \qquad (6)$$

The hydrostatic assumption gives $p^*(x) = gH(x)$ and the dissipation rate $\varepsilon$ is based on the length scale $\eta$ and the mean turbulence level $\bar{q}^2$ in the mixing layer

$$\varepsilon = \beta \bar{q} q / \eta, \quad \beta \equiv \mathrm{const}. \qquad (7)$$

Note also that $\mathrm{sgn} u_y = \mathrm{sgn}(u^+ - u^-)$ in the mixing layer. Outside the mixing layer $u_y = 0$ and, as consequence, $u = u^-(x)$ for $0 < y < h^-(x)$; $u = u^+(x)$ for $H(x) - h^+(x) < y < H(x)$. The function $v(x)$ can be found in this region by the continuity equation from the boundary conditions $v|_{y=0} = 0$ and $v|_{y=H} = u^+ H_x$.
Therefore, at the boundaries of the mixing layer the velocities $u(x, y)$ and $v(x, y)$ are known and $q(x, y) = 0$. It is required to construct a continuous solution of (5) – (7) inside the mixing layer $(h^-(x) < y < H(x) - h^+(x))$.
Let $u_0^+ > u_0^-$ and the velocity profile be monotone $(u_y \geq 0)$. Then we have $\sigma \equiv \sigma_0$. It is convenient to use the variables $x$ and $\psi$ as independent variables ($\psi$ is the stream function). In this variables (5) becomes a semilinear system

$$\begin{aligned} u_x - \sigma \bar{q} q_\psi &= -g H_x / u \\ q_x - \sigma \bar{q} u_\psi &= -\beta \bar{q} q / (\eta u). \end{aligned} \qquad (8)$$

A solution of (8) is constructed for $0 \leq x \leq x_1$, $\psi_0^- \leq \psi \leq \psi_0^+$, where

$$\psi_0^- = -\int_0^{h_0} u(0, y)\, dy = -h_0^- u_0^-,$$

$$\psi_0^+ = \int_{h_0}^{H_0} u(0, y)\, dy = h_0^+ u_0^+.$$

The point A, at which two uniform layers with different velocities merge, corresponds to the origin of coordinates on the $(\psi, x)$-plane (Fig. 2). The boundaries of the mixing layer are represented by the curves AB and AC. The solution $u = u^-(x)$, $q = \bar{q} = 0$ is known to the left of AB (region I). Similarly, the solution of (8) has the form $u = u^+(x)$, $q = \bar{q} = 0$ to the right of AC (region II).
Note that the curves AB and AC, which are given by the functions $\psi = \psi^-(x)$ and $\psi = \psi^+(x)$, respectively, are the characteristics of (8)

$$\frac{d\psi^\pm(x)}{dx} = \pm \sigma \bar{q}(x).$$

For the semilinear hyperbolic system (8) $(\bar{q}(x) > 0)$ we have the Goursat problem, which can be solved by the standard method of characteristics. If $u^-(x) \geq u_{\min} > 0$ in region I $(0 < x < x_1)$ the estimate $u(x, \psi) \geq u_{\min} > 0$ holds in BAC for a monotone velocity profile and the solution of (8) is bounded for

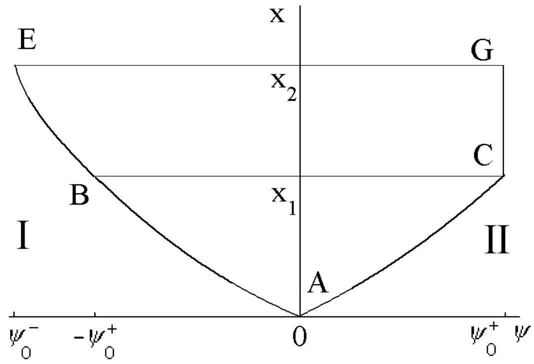


Figure 2: $(\psi, x)$-flow diagram

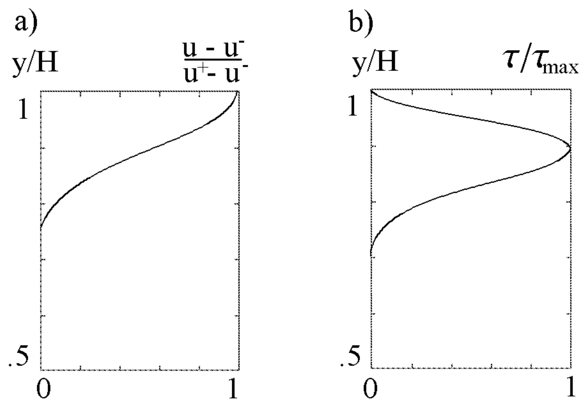


Figure3: a) Non-dimensional horizontal velocity $(u - u^-)/(u^+ - u^-)$ versus $y/H$ at $x = x_1$. b) Non-dimensional shear stress $\tau/\tau_{\max}$ versus $y/H$ at $x = x_1$. The initial flow parameters are given in figure 1 $(\theta = 0,\ \beta = 1.8)$.

$0 < x < x_1$. The behaviour of the solution near the origin of coordinates is represented by a self – similar solution of (8) for pressure – gradient – free flows $(p_x^* \equiv 0)$, which can be found in an explicit form [11]. It is shown in [11] that the energy conservation law $(\theta = 0)$ may be applied to describe the large billow evolution in mixing layers. In this case $\beta = 1.8$ gives an appropriate velocity and turbulence distribution in a self – similar mixing layer. Figure 3 shows the horizontal velocity (a) and the Reynolds stress (b) distribution in the mixing layer at the free surface $(x = x_1)$.

## 4. SURFACE JET

### 4.1. TWO-LAYER EQUATIONS

At $x = x_1$ we have the transition from the mixing layer to a turbulent surface jet (Fig. 1). As was mentioned above, system (1) describes the jet evolution with $h^+ \equiv 0$, $f^+ \equiv 0$. For steady-state flow the following relations are fulfilled

$$h^- u^- + Q = \bar{Q} = Q^+ + Q^- \quad (Q^+ < Q^-)$$
$$0.5{u^+}^2 + gH = J^- \qquad (9)$$
$$h^- {u^-}^2 + \eta \bar{u}^2 + 0.5gH = F.$$

All unknown variables can be found from (9) as functions of $Q = \eta \bar{u}$. Eqs (1) take the form

$$\frac{dQ}{dx} = \sigma_0 \bar{q}$$
$$\frac{d\bar{q}}{dx} = \frac{\sigma_0}{2Q}\Big(f_j(Q) - (1+2\delta)\bar{q}^2\Big), \qquad (10)$$

where the function $f_j(Q) = (u^- - \bar{u})^2$ is calculated from (9). Eqs. (10) may be reduced to the linear ODE

$$Q\frac{d\bar{q}^2}{dQ} = f_j(Q) - (1+2\delta)\bar{q}^2,$$

which has the solution

$$\bar{q}^2(Q) = \bar{q}^2(Q_1) + Q^{-(1+2\delta)} \int_{Q_1}^{Q} s^{2\delta} f_j(s)\ ds,$$

and the distribution of mean flow characteristics along the channel is calculated from the dependence

$$x = x_1 + \int_{Q_1}^{Q} \frac{ds}{\sigma_0 \bar{q}(s)}$$

up to the position $x = x_2$ where $Q = \bar{Q}$ (Fig.2). Analogously to the mixing layer, the free surface of the jet increases in subcritical flows ($\Delta_j > 0$) and decreases in supercritical flows ($\Delta_j < 0$) where

$$\Delta_j = 1 - \frac{gh^-}{{u^-}^2} - \frac{g\eta}{\bar{u}^2}.$$

The development of the surface jet is shown in Fig. 1 for dimensionless variables. The surface jet reaches the bottom at $x = x_2$.

### 4.2. VELOCITY FIELD IN SURFACE JET

Eqs. (5) – (7) describe also the velocity field in the surface jet ($x_1 < x < x_2$, $h^-(x) < y < H(x)$). As in the mixing layer, the flow at the lower boundary $y = h^-(x)$ of the jet is known ($u = u^-(x)$, $v = -h^- u^-_x$, $q = 0$). At the free boundary $y = H(x)$ we have $q = 0$ together with the kinematic condition. On the $(x, \psi)$-plane it gives the following initial–boundary problem for (8):

$$(u,q)\big|_{BC} = (u(x_1,y), q(x_1,y)),\ h^-(x) < y < H(x_1)$$
$$(u,q)\big|_{BE} = (u^-(x), 0),\ x_1 < x < x_2,\ q\big|_{CG} = 0. \quad (11)$$

The problem (5) – (7), (11) can be solved by the method of characteristics too.

## 5. CONCLUSIONS

The model of the two-dimensional turbulent layer evolution at free and rigid boundaries is considered. The model is applied here to the problem on interaction of the mixing layer with a free surface, but it can be used for a wide class of turbulent flows in open and closed channels with different sources of turbulence (breaking waves, floating and submerged bodies, topography etc.) The buoyancy effects due to the upper layer aeration may be incorporated in the model in a natural way. We hope that the two–level description of turbulent flow in the channels of finite depth may be useful for the more complicated cases such as surface or internal hydraulic jumps and bores with a zone of reverse flow at the fronts of them.

## 6. ACKNOWLEDGEMENTS

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