Sloshing problem in a half-plane covered by a dock with two equal gaps

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SUMMARY

The two-dimensional sloshing problem is considered in a half-plane covered by a rigid dock with two equal gaps. It is proved that the antisymmetric (symmetric) sloshing eigenvalues are monotonically decreasing (increasing) functions of spacing between gaps and formulae for their derivatives in terms of energy integrals are obtained. It is established that the eigenvalues are simple, and their asymptotics are found as spacing tends to infinity. Results are illustrated numerically.

1. INTRODUCTION

Sloshing problem in a half-plane covered by a rigid dock with a gap has received much consideration (see [2] and references cited therein) because eigenvalues of this problem furnish universal upper bounds for sloshing frequencies in the two-dimensional domains having the same free surface. Here we consider the case of a dock with two symmetric gaps and establish that the corresponding eigenvalues are simple and depend monotonically on spacing between gaps.

Let an inviscid, incompressible, heavy fluid occupy the half-plane $y < 0$ and be covered by a rigid dock so that the free surface consists of two gaps $\{b < |x| < b+1,\ y = 0\}$ (it is convenient to use non-dimensional Cartesian coordinates such that each gap has a unit length). Neglecting the surface tension, we consider free, small-amplitude, time-harmonic oscillations of fluid and its motion is assumed to be irrotational. Since the fluid domain is symmetric about the y-axis, sloshing modes are either *symmetric* or *antisymmetric*, that is, are even or odd functions of x respectively, and so we restrict our considerations to the quadrant $\{x \geqslant 0,\ y \leqslant 0\}$.

2. ANTISYMMETRIC MODES

The boundary value problem for a time-independent, antisymmetric velocity potential $u^{(-)}(x,y)$ is as follows:

$$\nabla^2 u^{(-)} = 0, \qquad x > 0,\ y < 0, \tag{1}$$

$$u^{(-)} = 0, \qquad x = 0,\ y < 0, \tag{2}$$

$$u^{(-)}_y = 0, \qquad 0 < x < b,\ x > b+1,\ y = 0, \tag{3}$$

$$u^{(-)}_y - \nu^{(-)} u^{(-)} = 0, \qquad b < x < b+1,\ y = 0. \tag{4}$$

Solutions to (1)–(4) are sought in the natural class of functions having finite kinetic energy, that is,

$$\int_{-\infty}^{0} \int_{0}^{+\infty} \left|\nabla u^{(-)}\right|^2 \mathrm{d}x\, \mathrm{d}y < \infty. \tag{5}$$

This condition shows that $u^{(-)}$ can be assumed to be a real function. Moreover, (5) provides that $u^{(-)}$ is continuous up to the x-axis and $\nabla u^{(-)}$ has a logarithmic singularity at the dock tips. Our aim is to investigate properties of eigenvalues $\nu^{(-)}(b)$ and eigenfunctions $u^{(-)}(x,y;b)$ (sloshing modes) as functions of b, but, unless it is necessary, we do not indicate this dependence for the sake of brevity.

By virtue of

$$u^{(-)}(x,y) = \frac{1}{2\pi} \int_b^{b+1} w^{(-)}(\xi - b)\, \log \frac{(x+\xi)^2 + y^2}{(x-\xi)^2 + y^2}\, \mathrm{d}\xi, \tag{6}$$

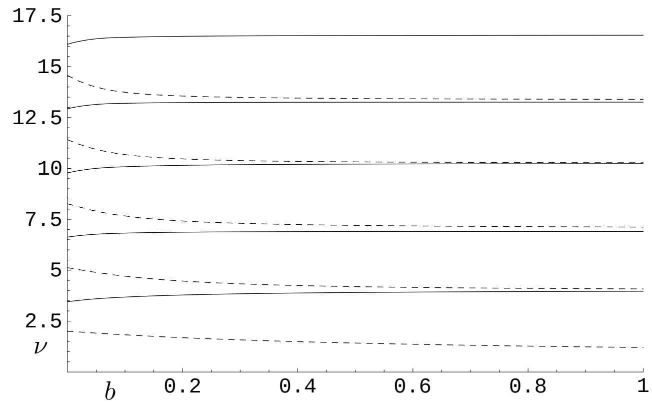

Figure 1: $\nu_1^{(-)}, \dots, \nu_5^{(-)}$ (dashed lines) and $\nu_1^{(+)}, \dots, \nu_5^{(+)}$ (solid lines) as functions of $b \in [0,1]$.

(1)–(5) is equivalent to the following spectral problem:

$$w^{(-)}(x) = \frac{\nu^{(-)}}{\pi} \int_0^1 \Big[\log(x+\xi+2b) - \log|x-\xi|\Big]\, w^{(-)}(\xi)\,\mathrm{d}\xi, \quad x \in (0,1), \tag{7}$$

where the integral operator $\boldsymbol{K}^{(-)}$ in the right-hand side is a compact, selfadjoint, positive operator in $L_2(0,1)$. In order to prove the equivalence of (1)–(5) and (7), one has to use properties of the single layer potential (6). The null-space of $\boldsymbol{K}^{(-)}$ is trivial.

Applying known results for weakly singular, selfadjoint, positive, integral operators including Jentsch's theorem (see, for example, [6], sect. 20), we find that for a fixed $b \geqslant 0$, there exists a sequence of eigenvalues

$$0 < \nu_1^{(-)} < \nu_2^{(-)} \leqslant \dots \leqslant \nu_n^{(-)} \leqslant \dots \quad \text{such that} \quad \nu_n^{(-)} \to \infty \ \text{ as } \ n \to \infty;$$

the eigenvalue $\nu_1^{(-)}$ is simple and the corresponding eigenfunction $w_1^{(-)}(x)$ is continuous and non-negative in $[0,1]$. Further results on the simplicity of eigenvalues are formulated in sect. 4.

Since $\boldsymbol{K}^{(-)}$ depends on $b \geqslant 0$ continuously and the kernel of $\boldsymbol{K}^{(-)}$ is a monotonically increasing function of b, the known results (see, for example, [5], sect. 95) yield that for each n $(= 1, 2, \dots)$ either $\nu_n^{(-)}$ and $w_n^{(-)}(x)$ are continuous functions of $b \geqslant 0$, and $\nu_n^{(-)}$ decreases with b. Quantitatively, the rate of decreasing is characterized by the following formula (its proof is analogous to that of (12) below):

$$\frac{\mathrm{d}\nu_n^{(-)}}{\mathrm{d}b} = -\int_{-\infty}^0 \left|\partial_x\, u_n^{(-)}(0,y)\right|^2 \mathrm{d}y \Bigg/ \int_b^{b+1} \left|u_n^{(-)}(x,0)\right|^2 \mathrm{d}x, \quad b > 0 \quad n = 1, 2, \dots$$

Fig. 1 shows that for $n = 2, 4$ the values $\nu_n^{(-)}(b)$ obtained from (7) (they are shown by dashed lines) are sufficiently close to the limit values even for $b = 1$, and the limit values are equal to the antisymmetric sloshing eigenvalues in a half-plane covered by a dock with a single gap of unit length (the latter eigenvalues are given in [2]). This means that if the spacing between gaps is sufficiently large, then fluid oscillations in each gap take place as if there is no other gap.

3. SYMMETRIC MODES

A real velocity potential $u^{(+)}(x,y)$ of symmetric sloshing mode satisfies the same conditions (1) and (3)–(5) as $u^{(-)}(x,y)$, and the parameter $\nu^{(+)}$ replaces $\nu^{(-)}$ in (4), but

$$u_x^{(+)} = 0, \quad x = 0,\ y < 0, \tag{8}$$

must be imposed instead of (2). There is a trivial symmetric sloshing mode $u_0^{(+)}(x,y) \equiv 1$ corresponding to $\nu_0^{(+)} = 0$. It follows from Green's formula that non-trivial symmetric eigensolutions satisfy

$$\int_b^{b+1} u^{(+)}(x,0)\,\mathrm{d}x = 0. \tag{9}$$

The following representation:

$$u^{(+)}(x,y) = -\frac{1}{2\pi}\int_b^{b+1} w^{(+)}(\xi-b)\,\log\Big[\big((x+\xi)^2+y^2\big)\big((x-\xi)^2+y^2\big)\Big]\,\mathrm{d}\xi, \tag{10}$$

which is similar to (6), leads to the spectral problem:

$$w^{(+)}(x) = -\nu^{(+)}\pi^{-1}\int_0^1 \Big[\log(x+\xi+2b)+\log|x-\xi|\Big]\,w^{(+)}(\xi)\,\mathrm{d}\xi, \quad x\in(0,1).$$

Unfortunately, it has no solution satisfying (9). The fact is that a function given by (10) and satisfying (9) decays at infinity. At the same time, any symmetric eigenmode must have a non-zero limit as $|z|\to\infty$.

Nevertheless, a spectral problem involving an integral operator with more complicated kernel can be obtained in the present case. The starting point is the function

$$W(z;\xi) = -\frac{1}{\pi}\left\{\log\frac{4(z-\xi)}{(1-2z)(1-2\xi)} - \frac{1+2z}{2}\log\left(\frac{1+2z}{1-2z}\right) - \frac{1+2\xi}{2}\log\left(\frac{1+2\xi}{1-2\xi}\right) + \frac{1}{2} - \pi\mathrm{i}(z+\xi)\right\}$$

($z=x+\mathrm{i}y$). It was derived in [1], where $\operatorname{Re} W(x,\xi)$ appeared as the kernel in the integral equation equivalent to sloshing problem for the dock with the single gap $\{-1/2<x<+1/2,\ y=0\}$.

We introduce Green's function

$$\begin{aligned} G(x,y;\xi) = 2^{-1}\operatorname{Re}\big\{&W(z+b+1/2;\xi+b+1/2)+W(z-b-1/2;\xi-b-1/2)\\ &+W(z+b+1/2;-\xi+b+1/2)+W(z-b-1/2;-\xi-b-1/2)\\ &-\pi^{-1}\big[2b^2\log(2b)+2(1+b^2)\log[2(1+b)]-(1+2b)^2\log(1+2b)\big]\big\}, \end{aligned}$$

which satisfies (1), (3), (5) and (8) as a function of (x,y); besides, $\int_b^{b+1} G(x,0;\xi)\,\mathrm{d}\xi=0$. Then, by virtue of

$$u^{(+)}(x,y) = \int_b^{b+1} w^{(+)}(\xi-b)\;G(x,y;\xi)\,\mathrm{d}\xi,$$

sloshing problem for non-trivial symmetric modes is equivalent to the following spectral problem:

$$w^{(+)}(x) = \nu^{(+)}\int_0^1 G(x+b,0;\xi+b)\;w^{(+)}(\xi)\,\mathrm{d}\xi, \quad x\in(0,1), \tag{11}$$

where the integral operator in the right-hand side is a compact, selfadjoint operator in the subspace of $L_2(0,1)$ consisting of functions which satisfy (9).

As in sect. 2, this yields that there exists a sequence of eigenvalues

$$0<\nu_1^{(+)}\leqslant\ldots\leqslant\nu_n^{(+)}\leqslant\ldots \quad \text{such that} \quad \nu_n^{(+)}\to\infty \ \text{ as } \ n\to\infty;$$

for each $n=1,2,\ldots$, either $\nu_n^{(+)}$ and $w_n^{(+)}(x)$ are continuous functions of $b\geqslant 0$ and these functions are analytic for $b>0$. The main result of the present section is the following identity:

$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} = \int_{-\infty}^0 \big|\partial_y u_n^{(+)}(0,y)\big|^2\,\mathrm{d}y \Big/ \int_b^{b+1}\big|u_n^{(+)}(x,0)\big|^2\,\mathrm{d}x, \quad b>0, \quad n=1,2,\ldots, \tag{12}$$

which implies that $\nu_n^{(+)}$ is a monotonically increasing function of $b\geqslant 0$.

Proof. Let $u_n^{(+)}(x,y;b)$ be a symmetric eigenmode corresponding to the sloshing eigenvalue $\nu_n^{(+)}(b)$. The above results imply that $\nu_n^{(+)}(b)$ is a differentiable function of $b>0$. Let Δ be a sufficiently small number (such that $b+\Delta>0$). After extending $u_n^{(+)}(x,y;b+\Delta)$ to the whole half-plane $y<0$ by means of the Schwarz Reflection Principle, we consider $u_n^{(+)}(x+\Delta,y;b+\Delta)$ defined in the closed quadrant $\{x\geqslant 0,\ y\leqslant 0\}$ even when $\Delta<0$. The latter function satisfies the similar boundary conditions as $u_n^{(+)}(x,y;b)$ on $\{0<x<b,\ y=0\}$, $\{b<x<b+1,\ y=0\}$ and $\{b+1<x<+\infty,\ y=0\}$, respectively. The second Green's formula for

$u_n^{(+)}(x, y; b)$ and $u_n^{(+)}(x+\Delta, y; b+\Delta)$ in $\{x > 0,\ y < 0\}$ gives

$$\int_b^{b+1} \Big[u_n^{(+)}(x,0;b)\, \partial_y\, u_n^{(+)}(x+\Delta,0;b+\Delta) - u_n^{(+)}(x+\Delta,0;b+\Delta)\, \partial_y\, u_n^{(+)}(x,0;b) \Big] \, \mathrm{d}x$$
$$= \int_{-\infty}^{0} \Big[u_n^{(+)}(0,y;b)\, \partial_x\, u_n^{(+)}(\Delta,y;b+\Delta) - u_n^{(+)}(\Delta,y;b+\Delta)\, \partial_x\, u_n^{(+)}(0,y;b) \Big] \, \mathrm{d}y$$

because (5) guarantees that the integral over a large quarter-circle tends to zero as its radius goes to infinity; the homogeneous Neumann condition on the dock is also applied here. Using (8) and the Lagrange theorem in the second integral, and the free surface conditions in the first one, we get

$$\Big[\nu_n^{(+)}(b+\Delta) - \nu_n^{(+)}(b) \Big] \int_b^{b+1} u_n^{(+)}(x,0;b)\ u_n^{(+)}(x+\Delta,0;b+\Delta)\, \mathrm{d}x$$
$$= \Delta \int_{-\infty}^{0} u_n^{(+)}(0,y;b)\, \partial_x^2\, u_n^{(+)}(\theta(y)\Delta, y; b+\Delta)\, \mathrm{d}y,$$

where $0 < \theta(y) < 1$ for $y \in (-\infty, 0)$. Letting $\Delta \to 0$ in this equation divided by Δ produces

$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} \int_b^{b+1} \big| u_n^{(+)}(x,0;b) \big|^2 \, \mathrm{d}x = \int_{-\infty}^{0} u_n^{(+)}(0,y;b)\, \partial_x^2\, u_n^{(+)}(0,y;b)\, \mathrm{d}y.$$

In order to obtain (12), it remains to transform the last integral using the Laplace equation and then applying integration by parts. The integrated terms vanish because $\partial_y\, u_n^{(+)}(0,y;b)$ satisfies the no flow condition on the dock and decays at infinity.

As in the antisymmetric case, numerical computations show that for large b the value of $\nu_n^{(+)}(b)$ obtained from (11) asymptotes the n-th sloshing eigenvalue in a half-plane covered by a dock with a single gap of unit length. Fig. 1 shows that for $n = 1, 3, 5$ the values $\nu_n^{(+)}(b)$ (they are shown by solid lines) are sufficiently close to the described limit values even for $b = 1$.

4. SIMPLICITY OF EIGENVALUES

Here we formulate the following result.

All symmetric eigenvalues are simple for any $b \geqslant 0$. For antisymmetric eigenvalues this property holds at least for $b = 0$ and sufficiently small positive b.

To the authors' knowledge, there are only two papers treating the question of simplicity of the sloshing eigenvalues. In [3], it is demonstrated that the first eigenvalue is simple, and a condition guaranteeing that the second eigenvalue is simple is obtained in [4].

5. ACKNOWLEDGEMENT

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Discussion Sheet

Abstract Title :	Sloshing problem in a half-plane covered by a dock with two equal gaps		
(Or) Proceedings Paper No. :	35	***Page :***	135
First Author :	Motygin, O.V.		
Discusser :	Maureen McIver		

Questions / Comments :

You say that there are no symmetric modes which decay to zero at infinity. Do the modes which you have found have finite energy?

Author's Reply :
(If Available)

Indeed, a symmetric mode has a non-zero limit at infinity; the asymptotics is as follows: $U_n^{(+)}$ = const + 0 ($|Z|^{-2}$) as $|Z| \to \infty$ and the formula can be differentiated therefore, the kinetic energy $\int\int_{\Re^2} \left|\nabla U_n^{(+)}\right|^2 dx\,dy$ is finite.