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# Wave Trapping by axisymmetric concentric cylinders

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**Summary**

The question of the uniqueness of solutions to the linearised water wave equations was settled once and for all in a paper in the Journal of Fluid Mechanics in 1996 by Dr. Maureen McIver. She constructed a solution for the motion between a pair of fixed rigid surface-piercing cylinders in two dimensions which decayed at large distances from the cylinders. Soon after she was joined by Professor P. McIver in producing an axisymmetric example in the form of a fixed rigid surface-piercing toroid of a special shape which supported an oscillatory motion in its interior fluid region whilst the motion in the exterior region decayed to zero. This wave trapping effect or non-uniqueness occurred for a particular relation between the wave frequency and the toroid geometry. In the present paper we show that such a phenomenon can occur for simple geometries also. In particular we show that wave trapping can occur in the annular region between two partially immersed vertical concentric circular cylindrical shells for particular values of radii and frequencies.

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## 1. Introduction

Classical linear water wave theory has been established for over one hundred and fifty years, yet fundamental questions remain concerning the conditions under which the governing equations are unique.

Over the years many eminent mathematicians have produced partial results proving uniqueness under certain geometrical assumptions concerning the shapes of the bodies present. Perhaps most notable of these is the work of John[2] who showed that the fluid motion produced by the oscillations of a partially-immersed axisymmetric non-bulbous cylinder with vertical sides at the free surface is unique. A corresponding result holds for the two-dimensional case of the motion of a cylinder section where the motion takes place in the plane of the section only. Both two and three-dimensional uniqueness results are also true for the scattering of a given incident wave field by the cylinder section or axisymmetric cylinder.

The question of the uniqueness or otherwise of the two-dimensional water wave problem was finally put to rest recently when Maureen McIver[4] produced a solution describing a localised oscillation between two particular cylindrical sections intersecting the free surface and having an internal free surface between them. Any multiple of such a solution could be added to either the scattering or radiation problem, rendering that problem non-unique. Further results followed rapidly. Particularly noteworthy was the extension to three dimensions by McIver & McIver[6] who constructed certain partially-immersed toroids having the property that oscillations at a certain frequency could exist in the internal water region bounded by the free surface and the toroid which did not radiate energy to infinity. A paper by Phil McIver and Nick Newman[7] at the last workshop showed how trapped modes could also be found for non-axisymmetric partially-submerged bodies.

Arguments in favour of there being a non-uniqueness in two dimensions have been expressed for some time by one of the present authors (DVE) who showed (Evans & Morris [1]) that a pair of partially immersed thin vertical barriers could, for certain geometries and wave frequencies, totally reflect an incident wave. He argued that if an identical pair of barriers was positioned a particular (large) distance from the first, then the totally reflected waves from one pair would be totally reflected on reaching the second, resulting in a standing wave in the region between the pairs of barriers with no waves outside. The correctness of this heuristic argument was confirmed recently by Kuznetsov et.al.[3] who utilised a powerful Galerkin method first described in the water-wave context by Porter & Evans[8] to consider the two-barrier problem numerically. They were able to find trapped modes between the pairs of barriers at geometries and frequencies close to those predicted by the wide-spacing argument described above. Indeed the approximate values provided an invaluable starting point for determining the actual accurate values for the four-barrier problem.

In the present work we have considered the axisymmetric version of the four barrier problem, namely, two concentric partially-immersed open-ended circular cylindrical shells in finite depth water. We find trapped modes in the form of local oscillations of the fluid in the two internal regions, namely that between the shells and the interior region bounded at the free surface by the smaller cylinder, which do not radiate energy into the outer region, external to the larger cylinder. Motivated in our search for such trapped modes by the work described at the last Workshop by Maureen McIver & Richard Porter[5] who extended the two-dimensional wide-spacing argument of Evans described above to the axisymmetric case of a submerged toroid. They argued that at high frequency the cylindrical waves in the region interior to the toroid could be approximated by equivalent plane waves encountering a local two-dimensional cylinder section which for some frequencies and geometry would totally reflect them. Thus an approximation to the trapped mode frequencies could be obtained using a quasi-two-dimensional approach. This approximation then provides the initial starting point for the search for trapped modes in the full axisymmetric problem. In §2. below we shall utilise the same plane-wave approximation, but generalised to higher angular modes.

## 2. Plane wave approximation

McIver & Porter[5] decribe a plane wave approximation for the fundamental axisymmetric standing wave in the interior of a torus. They show that for a torus of arbitrary cross section and radius $B$, trapped mode solutions occur when

$$kB = \frac{\pi}{4} - \frac{\delta}{2} + n\pi, \tag{1}$$

where $k$ is the wavenumber being the unique positive root of $\omega^2 = gk \tanh kh$ and $\omega$ id the radial frequency, $n$ is an arbitrary positive integer and $\delta$ is the phase of the reflection coefficient corresponding to a zero of transmission by the corresponding two-dimensional cylinder having the same cross-section as a section through the torus.
Here we extend the idea to include a $\theta$-variation in the standing wave. Thus, within our inner cylinder we consider a standing wave of the form

$$\phi = J_q(kr) \cos q\theta \psi_0(z), \tag{2}$$

where $J_q$ is the $q$th-order Bessel function of the first kind and $\psi_0(z) = N_0^{-1/2} \cosh k(z + h)$ with $N_0 = (1 + \sinh 2kh/2kh)/2$. Using similar arguments to those of McIver & Porter we find the possible trapped mode values for $kB$ are given by

$$kB = \frac{\pi}{4} - \frac{\delta}{2} + \frac{(2n + q)\pi}{2}, \tag{3}$$

where now $q$ indicates the $q$th angular mode. This relation can be seen to agree with (1) for the axisymmetric case when $q = 0$.

## 3. General formulation

Cylindrical polar coordinates are chosen with the $(r, \theta)$-plane in the undisturbed free surface and $z$ vertically upwards. The water is of depth $h$. Two thin concentric cylinder shells are placed in the water intersecting the free surface. The inner cylinder, which has radius $b_1$, is immersed to a depth $c_1$ and the outer cylinder of radius $b_2$ ($b_1 < b_2$), to a depth $c_2$. A time harmonic velocity potential of the form $\phi(r, \theta, z)e^{-i\omega t}$ is assumed such that the solution has angular frequency $\omega$. Then in the fluid the reduced velocity potential $\phi$ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{4}$$

Further $\phi$ must satisfy the no flow conditions

$$\phi_r = 0 \qquad \text{on } r = b_1 \text{ and } r = b_2, \tag{5}$$

$$\phi_z = 0 \qquad \text{on } z = -h, \tag{6}$$

and the free-surface condition

$$\phi_z - K\phi = 0 \tag{7}$$

where $K = \omega^2/g$.

We may write $\phi$ as a sum of $\theta$-dependent modes, in the form

$$\phi(r, \theta, z) = \sum_{q=0}^{\infty} \cos q\theta \chi_q(r, z). \tag{8}$$

whence from (4) the $\chi_q$ satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \chi_q}{\partial r}\right) + \frac{\partial^2 \chi_q}{\partial z^2} - \frac{q^2}{r^2}\chi_q = 0 \tag{9}$$

together with (7).
The method of solution of equations (4)-(9) involves expansions in eigenfunctions appropriate to the three fluid regions separated by the cylinders at $r = b_1$ and $r = b_2$. Continuity of pressure and horizontal velocity across $r = b_1$ and $r = b_2$, together with the condition that $\phi \to 0$ as $r \to \infty$ results in a pair of coupled homogeneous first-order integral equations for the unknown horizontal velocities at $r = b_1$ and $r = b_2$. Following a standard

(a)

| PWA | | Exact | |
|---|---|---|---|
| $b_1/c_1$ | $Kc_1$ | $b_1/c_1$ | $Kc_1$ |
| 3.559 | 1.1361 | 3.443 | 1.1452 |
| 6.324 | 1.1361 | 6.258 | 1.1393 |
| 9.089 | 1.1361 | 9.042 | 1.1378 |
| 1.527 | 2.5745 | 1.471 | 2.6071 |
| 2.747 | 2.5745 | 2.713 | 2.5871 |
| 3.967 | 2.5745 | 3.943 | 2.5812 |
| 5.188 | 2.5745 | 5.168 | 2.5787 |
| 6.408 | 2.5745 | 6.392 | 2.5774 |
| 7.627 | 2.5745 | 7.615 | 2.5765 |
| 8.848 | 2.5745 | 8.837 | 2.5760 |

(b)

| PWA | | Exact | |
|---|---|---|---|
| $b_1/c_1$ | $Kc_1$ | $b_1/c_1$ | $Kc_1$ |
| 2.176 | 1.1361 | 1.652 | 1.2011 |
| 4.942 | 1.1361 | 4.735 | 1.1504 |
| 7.706 | 1.1361 | 7.572 | 1.1427 |
| 0.917 | 2.5745 | 0.642 | 2.8719 |
| 2.137 | 2.5745 | 2.031 | 2.6273 |
| 3.357 | 2.5745 | 3.288 | 2.5976 |
| 4.577 | 2.5745 | 4.525 | 2.5874 |
| 5.798 | 2.5745 | 5.756 | 2.5829 |
| 7.018 | 2.5745 | 6.983 | 2.5804 |
| 8.237 | 2.5745 | 8.208 | 2.5788 |

Table 1: Exact and plane wave approximations to a selection of trapped mode solutions, with $s = 0.12$ and $c = 0.1$: (a) axi-symmetric($q = 0$) and (b) $q = 1$.

Galerkin approximation procedure (see for example, Kuznetsov et. al. [3]) these can be reduced to a homogeneous system of equations whose non-trivial solution, where it exists, indicates a trapped mode.

## 4. Results

The parameters of the problem are the wavenumber $k$, the cylinder radii $b_1$ and $b_2$ ($b_2 > b_2$), the depth of submergence of the cylinders $c_1$ and $c_2$ and the water depth $h$. To procede we fixed $c_1 = c_2 = c$, $s \equiv b_2 - b_1$ and the depth $h$ (assumed to be unity). By varying our other two variables $b_1$ and $k$ (using our approximation in (3) as a starting point), we are able to locate the particular geometries for which trapped mode solutions exist and the corresponding frequencies at which these trapped modes occur.
Tables 1(a) and 1(b) list some of the trapped mode geometries and wavenumbers for a particular value of $c$ and $s$, together with their approximate values as predicted by (3). The plane wave approximation used to derive (3) includes the assumption that $kB \ll 1$, thus we would expect our approximation to improve with larger $kb_1$. It can be seen from tables 1(a) and 1(b) that this is indeed true.
Also shown in Figure 1 are perspective views of selected trapped modes, together with their surface-profile cross-sections. If the water were to be, for example, 10m deep, then these trapped modes would exist within cylinders having depth of submergence 1m, differing in radius by 1.2m and in 1(a) have inner radius of 3.44m, the trapped mode having frequency 1.7Hz, in 1(b) have inner radius of 8.84m, the trapped mode having frequency 2.5Hz and 1(c) have inner radius of 8.21m, the trapped mode having frequency 2.5Hz.
The question remains as to how these trapped modes may be excited. Work is under way to determine if such resonances may be induced by the scattering of a normally incident plane wave having frequency close to the appropriate trapping frequency. Results for this problem will be presented at the Workshop.

## 5. References


1. Evans, D.V. & Morris, C.A.N. 1972 Complementary approximations to the solution of a problem in water waves. *J. Inst. Maths Applics.* **10**, 1–9.

2. John, F. 1950 On the motion of floating bodies, II. *Commun. Pure Appl. Maths* **3**, 45–101.

3. Kuznetsov, N., McIver, P. & Linton, C.M. 2001 On uniqueness and trapped modes in the water-wave problem for vertical barriers. *Wave Motion.* **33**, 283–307.

4. McIver, M. 1996 An example of non-uniqueness in the two-dimensional inear water wave problem. *J. Fluid Mech.* **315**, 257–266.

5. McIver, M. & Porter, R. 2001 Trapping of waves by a submerged elliptical torus. *Proc. 16th Int'l Workshop on Water Waves and Floating Bodies, Hiroshima, Japan.*

6. McIver, P. & McIver, M. 1997 Trapped modes in an axisymmetric water wave problem. *Q. J. Mech. Apll. Math.* **50**, 165–178.

7. McIver, P. & Newman, N. 2001 Non-axisymmetric trapping structures in the three-dimensional water wave problem. *Proc. 16th Int'l Workshop on Water Waves and Floating Bodies, Hiroshima, Japan.*

8. Porter, R. & Evans, D.V. 1995 Complementary approximations to wave scattering by vertical barriers. *J. Fluid Mech.* **294**, 155–180.

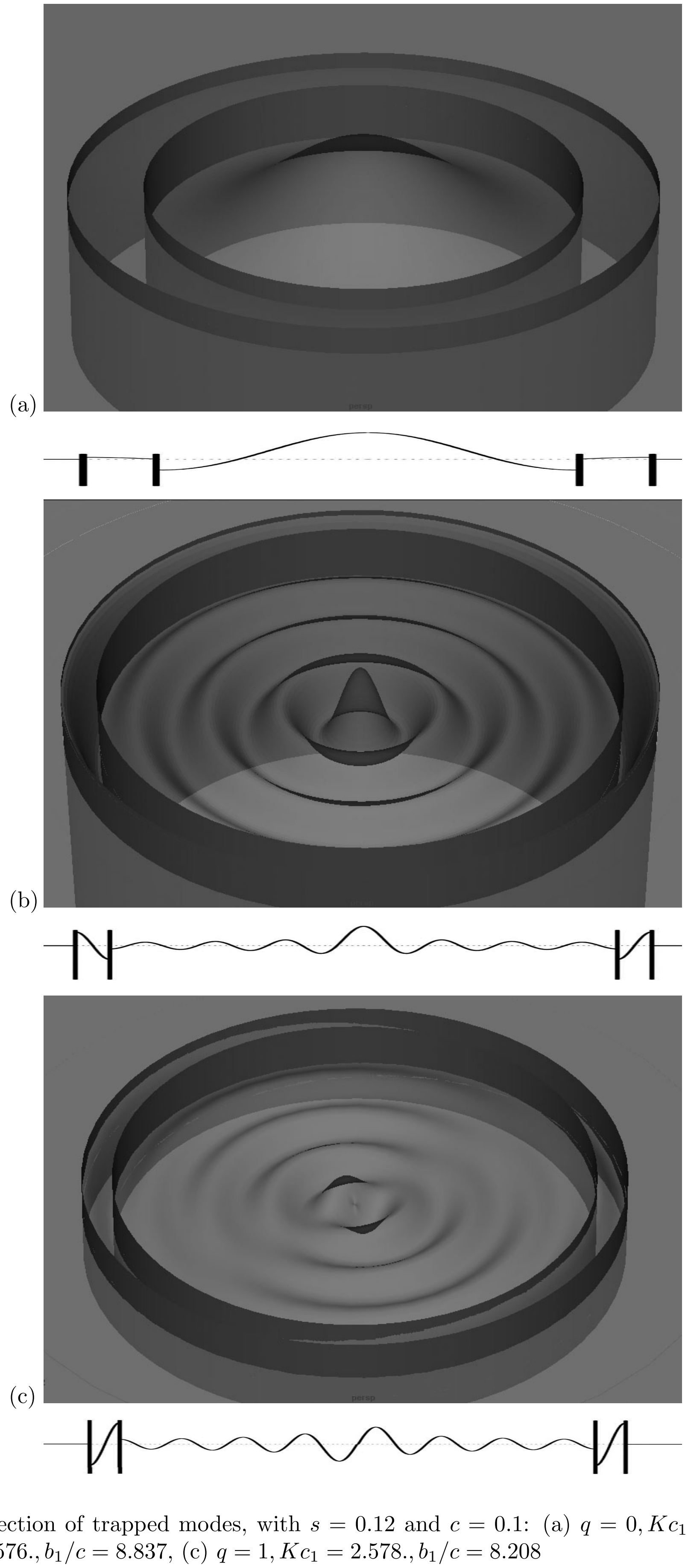

Figure 1: A selection of trapped modes, with $s = 0.12$ and $c = 0.1$: (a) $q = 0, Kc_1 = 1.145, b_1/c = 3.44$, (b) $q = 0, Kc_1 = 2.576., b_1/c = 8.837$, (c) $q = 1, Kc_1 = 2.578., b_1/c = 8.208$

# Discussion Sheet

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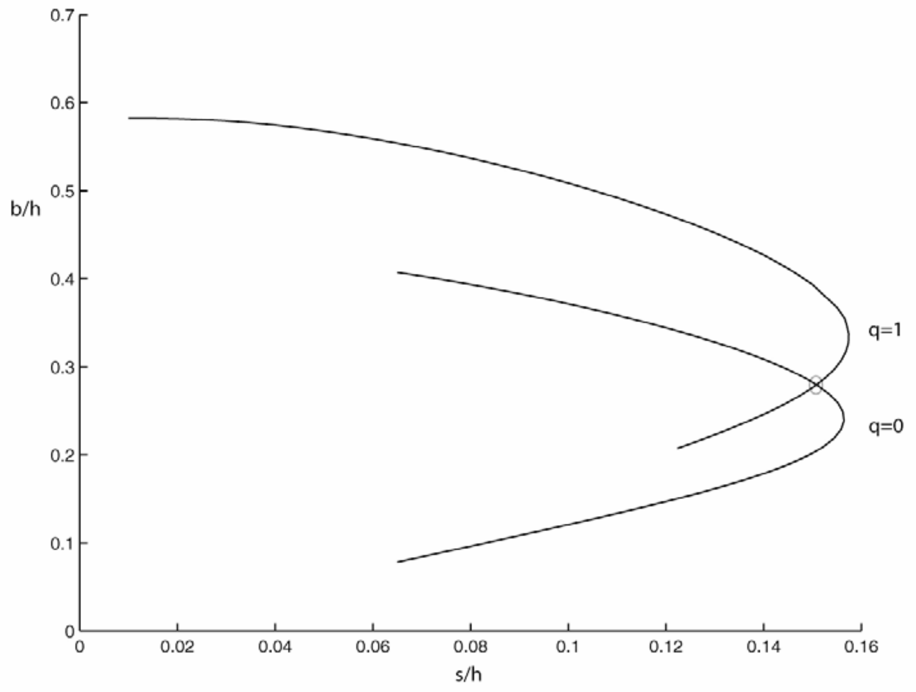
|---|---|
| **Abstract Title :** | Wave Trapping by axisymmetric concentric cylinders |
| ***(Or) Proceedings Paper No. :*** 42 | ***Page :*** 163 |
| **First Author :** | Shipway, B.J. |
| **Discusser :** | Maureen McIver |

**Questions / Comments :**

Are there enough parameters to vary to generate a single geometry for which there are 2 trapped modes (at different frequencies e.g. 1 axisymmetric and one with variation cosθ).

**Author's Reply :**
***(If Available)***

This is correct. As an example, the accompanying figure shows 2 curves of trapped mode geometries; one curve for an axisymmetric solution and one for a solution having a cos [theta] variation. In each case the submergence depth of the cylinders is 10% of the water depth. The crossing of these two curves thus indicates such a geometry supporting 2 trapped modes (though at different frequencies). I believe this to be the first instance of such a phenomenon in the water wave context.



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# Discussion Sheet

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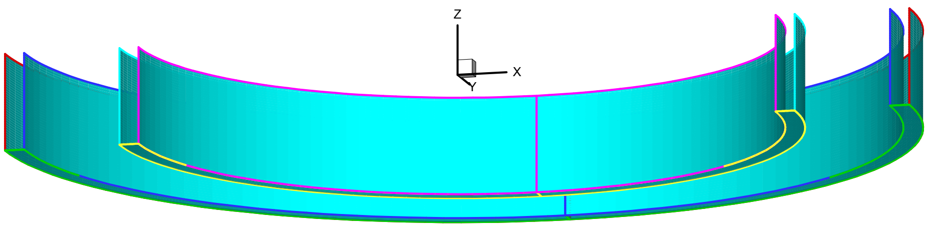
| **Abstract Title :** | Wave trapping by axisymmetric concentric cylinders | | |
| ***(Or) Proceedings Paper No. :*** | 42 | ***Page :*** | 163 |
| **First Author :** | B. J. Shipway | | |
| **Discusser :** | J. Nick Newman | | |

**Questions / Comments :**

This is a very interesting contribution, which may serve to indicate the broad range of possibilities for bodies which support trapped modes. Perhaps it is appropriate to speculate that the structures found so far are only the "tip of the iceberg".

One feature of the original toroids constructed by Maureen and Phil McIver is that these bodies coincided exactly with closed streamsurfaces. That provides a convincing argument for pure trapping. The present geometry is based on less precise numerical arguments which may leave open the possibility that these are `near-trapped modes'. I do not want to cast doubt on the present results, only to suggest that caution may be in order and further confirmation is desirable.

With the intention of providing some confirmation, I have made computations of the heave added-mass and damping using WAMIT. For the sake of simplicity I assumed that the fluid depth is infinite. Two slightly different geometries are analysed, with small but finite thickness of the cylinders. In both cases the cylinder depths are equal to 1.0, the radius of the inner free surface is 3.443, and the width of the outer annular free surface is 1.2, corresponding to the data shown in Table 1 for the first axisymmetric mode. In the first geometry, half of which is shown in perspective on the next page, the total thickness of each cylinder is 0.2. In the second geometry the total thickness is 0.1. The lower figure shows the resulting added mass and damping, based on quadratic B-spline representation of the potential and exact representation of the geometry. A total of 320 collocation points are used on each quadrant. The added mass is represented by the red double-signed "poles" and the damping by the black "delta functions", with the curves for thickness t=0.1 and t=0.2 identified separately. The character of these curves is essentially the same as for the toroid, as discussed in Reference 7 and in my 1999 paper on the axisymmetric toroid. The results shown here vary rapidly, over very narrow wavenumber bands, with peak magnitudes up to 1000. The singular wavenumbers (1.1386 for t=0.2, 1.1414 for t=0.1) are consistent with the value 1.1452 shown in Table 1 for zero thickness. Thus it seems reasonable to conclude that there is a high degree of confirmation, but the numerical approximation inherent in the WAMIT results prevents me from stating with certainty that pure trapping occurs based on these computations.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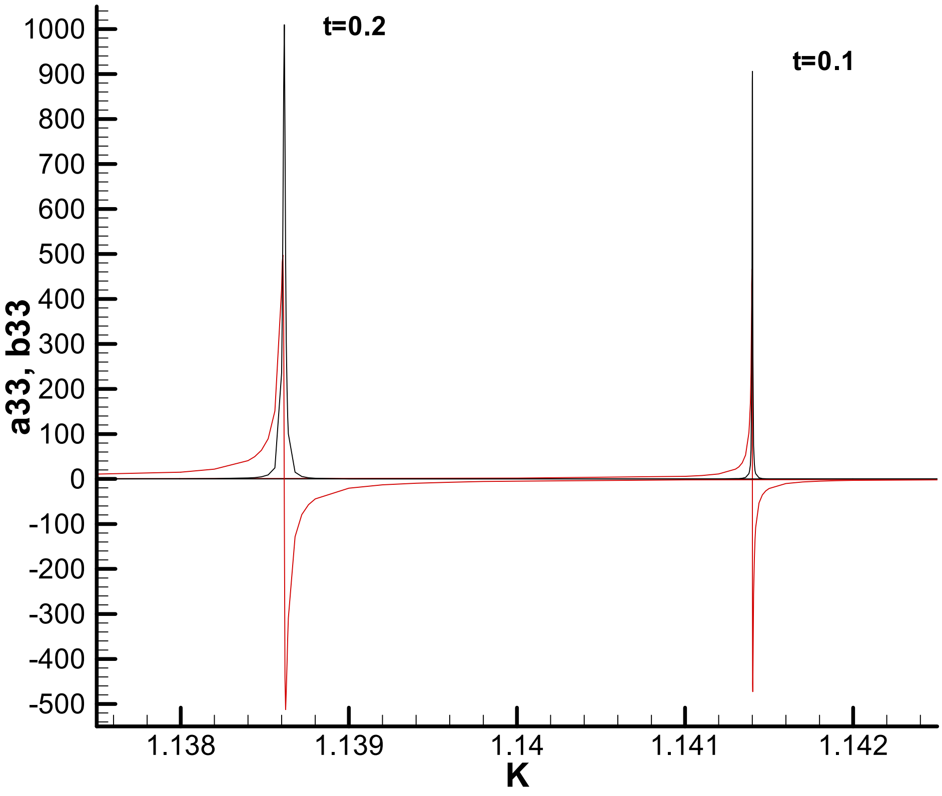
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**Author's Reply :**
***(If Available)***

It is gratifying that WAMIT has confirmed that the trapped modes do indeed appear to exist where predicted from our numerical approach. However it should be pointed out that the method we use, based on eigenfunction expansions, results in the need to solve singular integral equations which can be done very efficiently using a Galerkin procedure. The expansion functions are chosen to model the known singularities close to the edges of the shells giving very accurate results. Also the initial position of the trapped modes is facilitated by the wide spacing argument described in the Abstract.