STERN WAVES IN A CHANNEL. THE FORM OF THE WAVE MODES.

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1 The boundary value problem

For some time I have been working on the waves due to a submerged body moving with constant velocity U along a channel. Here I shall be concerned with just one aspect, the analytic form of the steady stern waves formed on the free surface behind the body. We take cartesian coordinates moving with the body, x horizontal across the channel, y vertical and increasing with depth, z horizontal along the channel. The mean free surface is at $y = 0$ and the side walls are at $x = \pm \ell$. For the sake of simplicity it is assumed that the motion is symmetrical about the mid-plane $x = 0$, but this restriction can be easily removed. It will then be expected that the wave modes can be found by separation of variables and will be of of the form $X(x)Y(y)Z(z)$, and this is indeed the correct result. I had expected to show this in the same way as for the Havelock wavemaker, but I encountered some difficulties which may be of interest to other workers in this field and which I shall describe here. The velocity potential is of the form

$$\Phi = Uz + \phi(x, y, z),$$

and it is assumed that the surface perturbation ϕ is so small that the free-surface condition of constant pressure can be linearized. Then the equation of continuity is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(x, y, z) = 0, \tag{1.1}$$

with the boundary conditions

$$\mu\frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial z^2} \quad \text{on} \quad y = 0, \quad \text{where} \quad \mu = g/U^2, \tag{1.2}$$

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{on} \quad x = \pm\ell, \text{ and } \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = h. \tag{1.3}$$

Since $\phi(x, y, z)$ is assumed to be an even function of x, we can write

$$\phi(x, y, z) = \frac{1}{2}\,\phi_0(y, z) + \sum_1^\infty \phi_m(y, z) \cos\frac{m\pi x}{\ell}. \tag{1.4}$$

Then

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\nu_m^2\right)\phi_m(y,z)=0,\ \text{where}\ \nu_m=\frac{m\pi}{\ell}, \tag{1.5}$$

with the boundary conditions

$$\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial}{\partial y}-\nu_m^2\right)\phi_m=0\ \text{ on }\ y=0,\ \text{and}\ \frac{\partial\phi_m}{\partial y}=0\ \text{ on }\ y=h. \tag{1.6}$$

2 The boundary value problem for $m > 0$

We now assume that the submerged body lies in the negative half-strip ($0 < y < h,\ -\infty < z < 0$). For the mth Fourier component we wish to find the form of the potential $\phi_m(y,z)$ satisfying the equation (1.5) in the semi-infinite strip ($0 < y < h,\ 0 < z < \infty$) , with the boundary conditions (1.6). Then by separation of variables we find that $\phi_m(y,z)$ has modal solutions of the form

$$\cosh Q_{mj}(h-y)\ \exp(ik_{mj}z). \tag{2.1}$$

Here Q_{mj} and k_{mj} satisfy

$$Q_{mj}^2-\nu_m^2=\mu Q_{mj}\tanh Q_{mj}h, \tag{2.2}$$

and

$$k_{mj}^2=Q_{mj}^2-\nu_m^2,$$

with a mode (2.1) corresponding to each solution Q_{mj} of the equation (2.2). Our principal result will be the following:

> **The potential $\phi_m(y,z)$ can be expressed as the sum of terms (2.1).**

(In the present section we shall henceforth omit the suffix m.) We might now try to proceed as in the familiar Havelock wavemaker theory, where the free-surface boundary condition is

$$\left(K+\frac{\partial}{\partial y}\right)\phi_m(y,z)=0\ \text{ on }\ y=0;$$

this theory depends on the completeness and orthogonality of the corresponding functions $\{\cosh q_j(h-y)\}$. For our set $\{\cosh Q_j(h-y)=f_j(y)\}$ it can be shown that the roots Q of the equation (2.2) are either real or pure imaginary, and that there is a quasi-orthogonality property

$$\int_0^h f_i(y)f_j(y)\,dy-\frac{1}{\mu}\,f_i(0)f_j(0)=0 \tag{2.3}$$

satisfied by the functions $f_i(y), f_j(y)$, when $Q_i^2 \neq Q_j^2$. We should note that the quadratic form

$$\int_0^h \{f(y)\}^2 \, dy - \frac{1}{\mu} \{f(0)\}^2$$

associated with (2.3) is not positive definite. However, we have not been able to show that the functions $\{f_j(y)\}$ are complete.

We therefore need to find a different argument to show that the potential $\phi(y,z)$ can be expressed in terms of the modes (2.1) only. (Only a brief outline can be given here.) An integral representation has been found for the potential $G_+(y,z;\eta,\zeta)$ of a submerged source

$$G_+(x,y;\xi,\eta) = \int_{-\infty}^{\infty} \exp(ikz)\frac{dk}{Q} \cosh Q(h-y)\frac{k^2 \sinh Q\eta - \mu Q \cosh Q\eta}{k^2 \cosh Qh - \mu Q \sinh Qh}, \quad (2.4)$$

where $Q^2 = k^2 + \nu^2$ and $\eta < y$. There are two real poles on the k-axis. If we assume that there are waves at $z = \infty$ and no waves at $z = -\infty$, the path of integration must be chosen to pass below these poles. It can be shown that G_+ can be expanded in terms of the modes (2.1).(This part of the argument is omitted.) We can then show that the expansion of $\phi(y,z)$ involves the same terms (2.1), by using a form of Green's representation involving sources and dipoles along the boundary. Let Green's identity be applied to $\phi(y,z)$ and $G_-(y,z;\eta,\zeta)$ in the semi-infinite strip $(0 < y < h, 0 < z < \infty)$, where the function ϕ is regular and the function G_- has a polar singularity $K_0(\nu r) = -\log(\frac{1}{2}\nu r) + O(1)$ near $(y,z) = (\eta,\zeta)$, and waves at $z = -\infty$. We find that

$$2\pi\phi(\eta,\zeta)$$
$$= \int_0^h \left\{\phi(y,0)\left(\frac{\partial}{\partial z}G_-(y,z;\eta,\zeta)\right) - \left(\frac{\partial\phi(y,z)}{\partial z}\right)G_-(y,0;\eta,\zeta)\right\}_{z=0} dy \quad (2.5)$$
$$- \int_0^\infty \left\{\phi(0,z)\left(\frac{\partial}{\partial y}G_-(y,z;\eta,\zeta)\right) - \left(\frac{\partial\phi(y,z)}{\partial y}\right)G_-(0,z;\eta,\zeta)\right\}_{y=0} dz \quad (2.6)$$

Using the surface boundary condition we find that the expression (2.6) is equal to

$$-\frac{1}{\mu}\int_0^\infty \left\{\phi(0,z)\frac{\partial^2}{\partial z^2}G_-(0,z;\eta,\zeta) - \frac{\partial^2\phi}{\partial z^2}G_-(0,z;\eta,\zeta)\right\}_{y=0} dz$$

$$= -\frac{1}{\mu}\left[\phi(0,z)\frac{\partial}{\partial z}G_-(0,z;\eta,\zeta) - \frac{\partial\phi}{\partial z}G_-(0,z;\eta,\zeta)\right]_{z=0}^{\infty}.$$

The contribution from $z = \infty$ vanishes because the terms involving G_- have no waves at $z = \infty$. Thus the potential can be represented by a continuous distribution of sources and dipoles over $(0 < y < h, z = 0)$, together with a discrete source and a discrete dipole at $(0,0)$. Here we use

$$G_+(y,z;\eta,\zeta) = G_-(\eta,\zeta;y,z). \tag{2.7}$$

As we have noted. these sources and dipoles have expansions in terms of the modes (2.1)

$$\cosh Q_j(h-y)\exp k_j z = \cos\{|Q_j|(h-y)\}\exp(-|k_j|z) \tag{2.8}$$

from the pure imaginary poles, together with wave terms from the real poles.

3 The case $m = 0$

When we try to apply the same argument to the potential $\phi_0(y,z)$ we meet with another difficulty. The source potential now becomes logarithmically infinite as $|z| \to \infty$, whereas for positive m it remains bounded at ∞. A different argument has therefore been used. The equation (1.5) is now the two-dimensional Laplace equation, and there is therefore a conjugate stream-function $\psi_0(y,z)$ satisfying the Laplace equation and the boundary conditions

$$\left(\frac{\partial^2}{\partial y \partial z} + \mu\frac{\partial}{\partial z}\right)\psi_0 = 0 \text{ on } y = 0, \quad \text{and } \frac{\partial \psi_0}{\partial z} = 0 \text{ on } y = h. \tag{3.1}$$

Thus

$$\left(\frac{\partial}{\partial y} + \mu\right)\psi_0 = c_0 \text{ on } y = 0, \text{ and } \psi_0 = c_h \text{ on } y = h, \tag{3.2}$$

where c_0 and c_h are constants. Clearly we can take $c_0 = 0$. If now we write

$$\Psi_0(y,z) = \psi_0(y,z) - \frac{\mu y - 1}{\mu h - 1}c_h,$$

then we observe that Ψ_0 satisfies Laplace's equation and the same boundary conditions (3.2) as ψ_0, but with $c_0 = 0$ and $c_h = 0$. This is a standard Sturm-Liouville problem, the eigenfunctions are the complete orthogonal set $\{\sinh k_j(h-y) = F_j(y)\}$, where $\mu \sinh k_j h - k_j \cosh k_j h = 0$. It can then be shown that $\Psi_0(y,z)$ can be expanded as the sum of modes

$$\sinh k_j(h-y)\exp(ik_j x)\ ,$$

and it follows that the conjugate potential $\Phi_0(y,z)$ can be expanded as the sum of modes

$$\cosh k_j(h-y)\exp(ik_j x)\ .$$

Analogous results can be found for infinite depth, involving integrals in place of series.