Second order directional wavemaker theory: prediction and control of free waves

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Abstract

This paper is devoted to a second order wavemaker theory for a three dimensional semi infinite basin. The solutions to the time harmonic boundary-value problems at first and second order are developed for regular waves. Potential at second order is decomposed into Stokes correction and free waves. Both components are given in a relatively simple form including progressive as well as evanescent first order interactions terms. The aim of this study is to precisely determine the amplitude of the free spurious waves in order to obtain a second order motion of the wavemaker able to suppress free wave generation. Then a second order correction of the wavemaker motion is deduced, which cancels the second order free waves.

Introduction

The theoretical analysis of wave generation is generally based on the assumption of potential flow in semi infinite basins. Most of previous works were concerned with two dimensional wave flumes. First the two dimensional linear wavemaker theory was developed, and showed a good agreement with experiments for small wavemaker stroke. However, for larger wavemaker motions, non-linear effects, including the generation of unwanted free waves were observed. These higher order effects motivated the development of second order theory with the aim of modeling the generation of free waves. Hudspeth and Sulisz [5] developed a complete second order solution for generic planar wavemakers. They noticed the importance of first order evanescent modes when calculating the second order free wave amplitude. They also showed that the time-independent second order solution predicted exactly the mean return flow in a closed wave flume. Their work was further extended to irregular wave generation by Schäffer [7].

Regarding wavemaker theory in three dimensions, only a few references are available. At first order, Biésel [1] derived the basic snake's principle, in which the basin is considered of infinite extent, without sidewalls. More sophisticated theories were further developed, see *e.g.* Dalrymple [4] or Boudet and Pérois [3], in which the influence of sidewalls is accounted for in order to maximise the spatial extent of the usable zone. At second order, Wu [9] developed a theory for directional regular waves, also called oblique waves. Investigations showed that free waves amplitude can be as large as Stokes bound second order waves. His work was extended by Suh and Dalrymple [8] to deal with a directional spectrum at first order. These two studies were based on the geometrical assumption of a basin infinite in the direction parallel to the wavemaker, and semi-infinite in the direction perpendicular to the wavemaker. At first order, however, it is well known that reflection on sidewalls will completely modify the wave field generated by the snake principle. Li and Williams [6] developed a complete second order theory in three dimensional basin with sidewalls. Like Hudspeth and Sulisz did for wave flumes, they present both the time dependent and the independent part of the solution at second order. Li and Williams' solution was again established for snake principle. In the present paper, a second order wavemaker theory is developed in three dimensional semi infinite basin with sidewalls. The time dependent second order solution is formulated for regular oblique waves. Resulting expressions for second order free waves are arranged in such a way that any first order generation technique can easily be accounted for. The aim of this study is to precisely determine the amplitude of the free spurious waves in order to obtain a second order motion of the wavemaker able to suppress free wave generation.

Theoretical development

We consider a rectangular wave basin of constant depth h and width b (see figure 1). The wavemaker is located at $x = 0$ and the basin is semi-infinite in the (Ox) direction. The y-axis is directed along the paddle and the

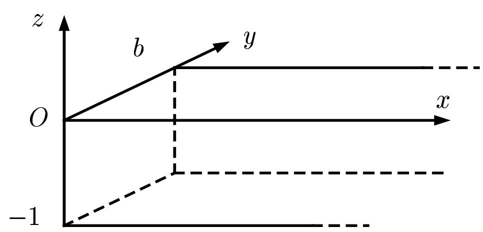

Figure 1: Basin sketch.

z-axis vertically upwards. The origin O of this (x, y, z) Cartesian coordinates system is chosen on a sidewall at still water level so that the two sidewalls are described

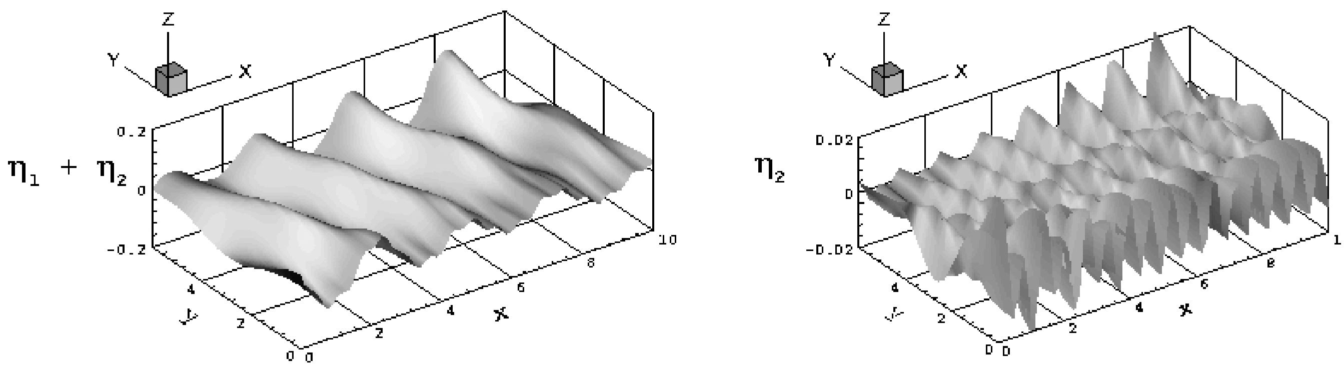

Figure 2: Analytical wave field, with $\lambda = 2.5$, $\theta = 20$ degrees. Left: first + second order, right: second order only (Dalrymple method in which $X_d = 3$).

by $y = 0$ and $y = b$ respectively. The fluid is considered inviscid, incompressible; the flow is irrotational. The problem is described in terms of the velocity potential ϕ. The classical perturbation method in combination with Taylor expansions of the boundary conditions at the free surface and at the wavemaker leads to a boundary-value problem for wave contributions at the first and second order in wave steepness. For simplicity, the development will further be written in complex representation and all quantities are nondimensionalised with respect to basin water depth h and acceleration g due to gravity. We are interested in the time harmonic solution when the prescribed wavemaker motion is sinusoidal in time. All quantities like ϕ, η and X may be expressed as real parts of complex values

$$\phi(x, y, z, t) = \text{Re}\left[\underline{\phi}(x, y, z)\, e^{i\omega t}\right]$$

where ω is the angular frequency of the wavemaker motion.

First order solution

The first order potential must satisfy Laplace equation everywhere, homogeneous Neumann conditions at the bottom and the sidewalls, an homogeneous free surface condition and an inhomogeneous Neuman condition at the wavemaker. A common potential solution is the double summation of spatial modes given by

$$\underline{\phi_1} = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{i\underline{a_{mn}}}{\omega} F(\alpha_m, z)\cos\mu_n y\, e^{-k_{mn}x} \quad (1)$$

$$F(\alpha, z) = \frac{e^{i\alpha(z+1)} + e^{-i\alpha(z+1)}}{e^{i\alpha} + e^{-i\alpha}}$$

where the wave numbers are defined by $\mu_n = \frac{n\pi}{b}$ with $n \in \mathbb{N}$; the $\{\alpha_o = ik,\ \alpha_m\}$ are the imaginary or real roots of the dispersion relationship $\omega^2 = i\alpha_m F(\alpha_m, 0)$ and finally the k_{mn} are given by $k_{mn} = \sqrt{\alpha_m^2 + \mu_n^2}$ if $\alpha_m^2 + \mu_n^2 > 0$ and $k_{mn} = i\sqrt{|\alpha_m^2 + \mu_n^2|}$ else.

Both progressive waves and evanescent modes appear in the potential expansion (1). The condition at the free surface $\underline{\eta_1} = \underline{\phi_1}_t$ for water elevation, gives for each mode a displacement $\eta = \underline{a_{mn}}\cos\mu_n y\, e^{-k_{mn}x}$, in which k_{mn} is purely imaginary or positive real for progressive waves or evanescent modes respectively. Progressive waves will correspond to the restricted domain $m = 0$ and $n \leq E\left[kb/\pi\right]$ (where $E\cdot$ is the integer value) to ensure that k_{mn} is purely imaginary and evanescent to all others couples of (m, n) subscripts. The amplitude $\underline{a_{mn}}$ of each potential mode may be obtained from the boundary condition at the wavemaker. Assuming that the position of the wavemaker X is described by $\underline{X}(y, z) = f(y)g(z)$ and using orthogonality of y- and z-modes, we get

$$\underline{a_{mn}} = -\frac{\omega^2(e^{i\alpha_m} + e^{-i\alpha_m})}{k_{mn}} J_m I_n$$

in which

$$J_m = \frac{\int_{-1}^{0} g(z)\left(e^{i\alpha_m(z+1)} + e^{-i\alpha_m(z+1)}\right) dz}{\int_{-1}^{0}\left(e^{i\alpha_m(z+1)} + e^{-i\alpha_m(z+1)}\right)^2 dz} \quad (2)$$

$$\text{and } I_n = \frac{\int_0^b f(y)\cos\mu_n y\, dy}{\int_0^b \cos^2\mu_n y\, dy} \quad (3)$$

This formulation is valid for all type of wavemaker and will lead to the potential provided the double summation converges at large m and n. Expressions (2) and (3) of J_m and I_n contain four integrals that can be easily calculated from wavemaker geometry and motion. Results are widely present in literature and omitted here for brevity.

In the method proposed by Dalrymple [4], the modal amplitudes are derived in an inverse way from a target potential

$$\underline{\phi}(X_d, y, z) = \frac{i\underline{a}}{\omega}\frac{\cosh k(z+1)}{\cosh k}\, e^{-ik(X_d\cos\theta + y\sin\theta)}$$

imposed at a distance X_d from the wavemaker. Hence we get

$$\underline{a_{on}} = \underline{a}\, I_n\, e^{(k_{on} - \alpha_o\cos\theta)X_d}$$

and the wavemaker motion can be deduced from these coefficients through the wavemaker condition

$$\underline{X_1}(y, z) = -g(z)\sum_{n=0}^{+\infty} \frac{k_{on}\,\underline{a_{on}}}{i\alpha_o\left(e^{i\alpha_o} - e^{-i\alpha_o}\right) J_o}\cos\mu_n y$$

A detailed study of the first evanescent mode with $m = 0$ and $n = N + 1$ reveals that its wave number tends to zero. Hence its characteristic vanishing length can reach great values and the mode will be visible in all the basin. Furthermore, this modal amplitude tends to infinity if the wave number goes to zero so that this mode must not be excited by the wavemaker. The simplest solution is to impose $\underline{a_{on}} = 0$ for all $n > N$ to avoid all such evanescent modes.

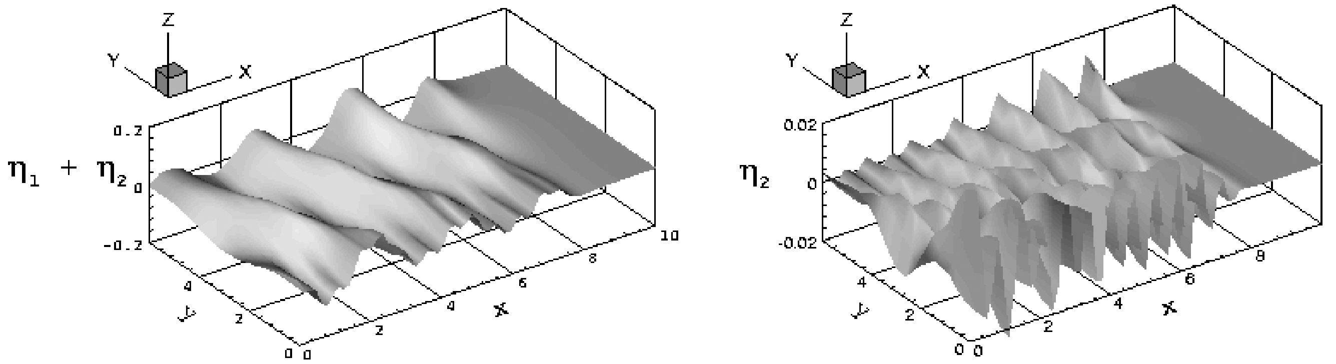

Figure 3: Time domain simulation, with $\lambda = 2.5$, $\theta = 20$ degrees, $t = 13T$. Left: first + second order, right: second order only (Dalrymple method in which $X_d = 3$).

Second order solution

The second order potential satisfies Laplace equation and homogeneous Neumann conditions at the bottom and sidewalls. The Neumann condition at the wavemaker is inhomogeneous, as well as the free surface condition. The previous first order solution appears in the right hand sides of these two conditions. A convenient way to solve these equations is to separate the second order potential into two parts, $\phi = \phi_s + \phi_f$ where the first term is the Stokes second order correction and the second one is the free wave part due to non linearities induced by the wavemaker. The Stokes correction will satisfy an inhomogeneous condition (4) at the free surface and no condition at the wavemaker (this condition will be satisfied later by the free wave component).

$$2i\omega\underline{\phi_e}+\underline{\phi_e}_z = -\frac{1}{2}2i\omega|\overrightarrow{\Delta}\underline{\phi_1}|^2-\frac{1}{2}\underline{\eta_1}[iw\underline{\phi_1}+\underline{\phi_1}_z]_z \text{ for } z=0 \quad (4)$$

As mentioned by Hudspeth and Sulisz [5], the right hand side of the above equation gives both a time dependent and a time independent solution for the potential. In the study of progressive waves we focus our attention on time dependent harmonic terms only. Here at second order, those terms will oscillate at frequency 2ω. The right hand side of equation (4) contains interaction terms involving wave-wave, wave-evanescent and both evanescent-evanescent components. The expression used to describe the first order potential allows us to treat all those interactions the same way. Hence, we obtain after some tedious algebra

$$\begin{aligned}\underline{\phi_s} \;=\;& \frac{i\underline{\underline{a^+_{mnpq}}}}{2\omega}\, e^{-(k_{mn}+k_{pq})x}\cos(\mu_n+\mu_q)y\; F(\alpha^+_{mnpq},z)\\ +\;& \frac{i\underline{\underline{a^-_{mnpq}}}}{2\omega}\, e^{-(k_{mn}+k_{pq})x}\cos(\mu_n-\mu_q)y\; F(\alpha^-_{mnpq},z)\end{aligned}$$

where the wave numbers are defined by $(\alpha^{\pm}_{mnpq})^2 = (k_{mn}+k_{pq})^2-(\mu_n\pm\mu_q)^2$ and the amplitudes

$$\underline{a^+_{mnpq}} \;=\; \frac{\underline{\underline{a_{mn}}}\,\underline{\underline{a_{pq}}}}{4}\,\frac{6\omega^4+2k_{mn}k_{pq}+(\alpha^+_{mnpq})^2-2\mu_n\mu_q}{-4\omega^2+i\alpha^+_{mnpq}\frac{e^{i\alpha^+_{mnpq}}-e^{-i\alpha^+_{mnpq}}}{e^{i\alpha^+_{mnpq}}+e^{-i\alpha^+_{mnpq}}}}$$

$$\underline{a^-_{mnpq}} \;=\; \frac{\underline{\underline{a_{mn}}}\,\underline{\underline{a_{pq}}}}{4}\,\frac{6\omega^4+2k_{mn}k_{pq}+(\alpha^-_{mnpq})^2+2\mu_n\mu_q}{-4\omega^2+i\alpha^-_{mnpq}\frac{e^{i\alpha^-_{mnpq}}-e^{-i\alpha^-_{mnpq}}}{e^{i\alpha^-_{mnpq}}+e^{-i\alpha^-_{mnpq}}}}$$

This solution is quite similar to the one of Li and Williams [6], with a correction of an apparent error in the denominators of $\underline{a^+_{mnpq}}$ and $\underline{a^-_{mnpq}}$.

The second potential $\underline{\phi_f}$ must satisfy the condition

$$\underline{\phi_f}_x = -\underline{\phi_s}_x - \underline{X_1}\,\underline{\phi_1}_{xx} + \underline{X_1}_y\underline{\phi_1}_y + \underline{X_1}_z\underline{\phi_1}_z \text{ for } x=0 \quad (5)$$

in order for the total potential to satisfy the right condition at the wavemaker. Like the first order potential, this second potential follows an homogeneous condition at the free surface. It is given in a similar way as the first order potential (1)

$$\underline{\phi_f} \;=\; \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{i\underline{a^f_{mn}}}{2\omega}\,F(\beta_m,z)\,\cos\mu_n y\; e^{-\gamma_{mn}x} \quad (6)$$

where the wave numbers are defined by $\mu_n = \frac{n\pi}{b}$ with $n \in \mathbb{R}$; the $\{\beta_o = ik_f, \beta_m\}$ are the imaginary or real roots of the dispersion relationship $4\omega^2 = i\beta_m\, F(\beta_m, 0)$ and finally the γ_{mn} are given by $\gamma_{mn} = \sqrt{\beta_m^2+\mu_n^2}$ if $\beta_m^2+\mu_n^2 > 0$, and $\gamma_{mn} = i\sqrt{|\beta_m^2+\mu_n^2|}$ else. This potential will thus in part correspond to progressive waves called free waves, which will eventually disturb the prescribed wave field. To annihilate these free waves, one may try to remove them by adding a second order wavemaker motion in phase opposition with the theoretical free waves. In the correction of the generation process, the first and most difficult step is to predict the free waves amplitude. This may be done with the inhomogeneous condition at the wavemaker (5). After some other tedious algebra, it comes from condition (5) at wavemaker

$$\begin{aligned}\underline{a^f_{mn}} = \frac{\beta_m}{\gamma_{mn}}\frac{(e^{i\beta_m}+e^{-i\beta_m})^2}{2\beta_m+\frac{e^{2i\beta_m}-e^{-2i\beta_m}}{2i}}\Bigg\{&\sum_{q=0}^{E[n/2]}(k_{m'n-q}+k_{pq})\\ &\frac{\underline{\underline{a_{m'n-q}}}\,\underline{\underline{a_{pq}}}}{2}\Bigg[\frac{6\omega^4+2k_{mn}k_{pq}+(\alpha^+_{mnpq})^2-2\mu_n\mu_q}{(\alpha^+_{m'n-q\,pq})^2-\beta_m^2}\\ &+\frac{\alpha_{m'}\alpha_p+k_{m'n-q}k_{pq}+\mu_{n-q}\mu_q}{2\alpha_{m'}\alpha_p}\frac{4\omega^4+(\alpha_{m'}-\alpha_p)^2}{(\alpha_{m'}+\alpha_p)^2-\beta_m^2}\\ &+\frac{\alpha_{m'}\alpha_p-k_{m'n-q}k_{pq}-\mu_{n-q}\mu_q}{2\alpha_{m'}\alpha_p}\frac{4\omega^4+(\alpha_{m'}+\alpha_p)^2}{(\alpha_{m'}-\alpha_p)^2-\beta_m^2}\Bigg]\\ &-A^{mn}_{m'p}+\frac{1}{1+\delta_{on}}\sum_{q=0}^{+\infty}(k_{m'n+q}+k_{pq})\frac{\underline{\underline{a_{m'n+q}}}\,\underline{\underline{a_{pq}}}}{2}\\ &\Bigg[\frac{6\omega^4+2k_{mn}k_{pq}+(\alpha^-_{mnpq})^2+2\mu_n\mu_q}{(\alpha^-_{m'n+q\,pq})^2-\beta_m^2}\end{aligned}$$

$$+\frac{\alpha_{m'}\alpha_p + k_{m'n+q}k_{pq} - \mu_{n+q}\mu_q}{2\alpha_{m'}\alpha_p}\frac{4\omega^4 + (\alpha_{m'} - \alpha_p)^2}{(\alpha_{m'} + \alpha_p)^2 - \beta_m^2}$$
$$+\frac{\alpha_{m'}\alpha_p - k_{m'n+q}k_{pq} + \mu_{n+q}\mu_q}{2\alpha_{m'}\alpha_p}\frac{4\omega^4 + (\alpha_{m'} + \alpha_p)^2}{(\alpha_{m'} - \alpha_p)^2 - \beta_m^2}\Bigg]\Bigg\}$$

where $A_{m'p}^{mn} = 0$ if n is odd or the term of the first finite summation with $q = n/2$ if n is even. One should notice that summation signs over m' and p have been omitted for brevity. It must be highlighted that this expression for the free wave amplitude is fully independent of the geometry of wavemaker. It means that this expression is valid for all types of wavemaker, provided the right first order amplitudes $\underline{a_{m'n\pm q}}\,\underline{a_{pq}}$ are known. The corresponding free surface elevation may be separated in the same way as the first order potential. Modes with $m = 0$ and $n \leq N_f = E\left[k_f b/\pi\right]$ will represent the free progressive waves whereas other modes will vanish far from the wavemaker.

Validations

In two dimensions, the free wave potential given by (6) reduces to the solution of Hudspeth and Sulisz [5]. Both Stokes and free wave potentials match exactly their solutions as the wave direction θ goes to zero.

Figure 2 presents three dimensional views of a regular wave field. The wave period is $T = 4$ and the wave direction is 20 degrees from main axis; the first order wavelength is $\lambda = 2.5$ and the wave steepness $2a/\lambda = 5\%$. The wave field is generated with the Dalrymple method for which the target distance is $X_d = 3$. The left plot shows the superposition of first and second order wave field. As expected from Dalrymple theory, one can observe a clean wave field between $x = 2$ and $x = 4$. After $x = 4$, waves will reach the sidewall at $y = 6$. Hence we can observe a reflection pattern in this region. The right plot represents the second order elevation. The Stokes correction and the free waves have been superposed for further comparison with simulation. Separate analysis shows that the large waves at the $y = 0$ wall on second order plot are free waves. These large free waves can be three times higher than the Stokes correction.

A nonlinear spectral model developed recently by Le Touzé *et al.* [2] has been employed to simulate three dimensional wave generation. This model is based on the same framework as the analytical solution presented here, but the problem is solved in the time domain, in a bounded basin. The boundary value problems at first and second order are solved using a spectral expansion of the potentials in series of the natural modes of the tank. Figure 3 presents numerical results for the same case as the previous analytical one. An absorbing zone is implemented in the simulation to avoid reflection on the back wall $x = 10$. The influence of this numerical beach is clearly visible on the two plots at $x > 8$ where the free surface elevation goes rapidly to zero. Comparisons between the two figures 2 and 3 shows that the simulation correctly resolves first and second order problems. The spectral numerical model thus represents a dedicated tool to investigate the range of validity of the analytical solution to predict free waves. Preliminary simulations in two dimension have shown the effectiveness of the second order wavemaker motion correction in suppressing free waves. Three dimensional simulations will be performed and results will be presented at the workshop.

Conclusion

A frequency domain second order analytical solution has been developed for wave generation and propagation in a three dimensional semi infinite basin. This second order wavemaker theory predicts unwanted free waves amplitudes. We are willing to use these predictions in numerical simulations in order to suppress free wave generation. Free waves predictions will also be applied to improve the wave generation process in the new offshore basin of Ecole Centrale de Nantes.

References

[1] F. Biésel. Wave machines. In *Proceedings of the First Conference on Ships and Waves*, 1954.

[2] D. Le Touzé F. Bonnefoy and P. Ferrant. Second-order spectral simulation of directional wave generation and propagation in 3d tank. In *Proceedings of the 12th Int. Offshore and Polar Engng Conf. ISOPE'02*, volume III, pages 173–179, Japan, 2002.

[3] L. Boudet and J.-P. Pérois. Nouvelles techniques de pilotage d'un batteur segmenté pour la génération de houle oblique. In *Comptes Rendus des Huitièmes Journées de l'Hydrodynamique*, ISSN 1161-1847, Nantes, France, 2001.

[4] R.A. Dalrymple. Directional wavemaker theory with sidewall reflection. *Journal of Hydraulic Research*, 27(1):23–24, 1989.

[5] R.T. Hudspeth and W. Sulisz. Stokes drift in two-dimensional wave flumes. *Journal of Fluid Mechanics*, 230:209–229, 1991.

[6] W. Li and A.N. Williams. Second-order waves in a three-dimensional wave basin with perfectly reflecting sidewalls. *Journal of Fluids and Structures*, 14(4):575–592, May 2000.

[7] H.A. Schäffer. Second-order wavemaker theory for irregular waves. *Ocean Engineering*, 23(1):47–88, 1996.

[8] K. Suh and R.A. Dalrymple. Directional wavemaker theory: a spectral approach. In *Proceedings of IARH Seminar*, pages 389–395, Lausanne, Swiss, 1987.

[9] Y.-C. Wu. *Directional wavemaker: theory and experiment.* PhD thesis, University of Delaware, 1985.