Capillary-gravity Waves due to an Impulsive Disturbance

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The solution to the classical Cauchy-Poisson problem of water waves generated by an impulsive disturbance under the pure-gravity effect (Lamb 1932) presents a perplexing peculiarity - the surface elevation in a region approaching to the impulsive disturbance (an initial elevation concentrated along a line of surface in 2D case) is found to diminish continuously in length and to increase continuously in height without limit. In 3D case, we consider the transient velocity potential generated by an impulsive source approaching to the free surface. It is remarked in Clément (1998) that the transient waves at a given instant oscillate with increasing amplitude and decreasing wavelength when we approach to the source point. Furthermore, the amplitude of transient waves in a given radial distance from the disturbance grow linearly with time while the wavelength decreases at the same rate. This peculiar property of pure-gravity waves hinders the numerical development to solve the boundary-value problem associated with a floating body in which the space integral over body's surface as well as the time-convolution integral are difficult to be accurate. The same situation is present in the frequency domain. The work on the wave pattern due to a steady-moving concentrated pressure on the free surface by Ursell (1960) showed that the wave elevation near the track of pressure point from the linearized *pure-gravity* theory oscillates with indefinitely increasing amplitude and indefinitely decreasing wavelength. For the more general case of a point source both *pulsating* and advancing at a uniform speed, the same behavior of the generated potential flow is revealed by Chen & Wu (2001) and described in an analytical way.

The surface tension is commonly ignored in describing water waves around large floating bodies, since its effect is considered to be significant only for rather short waves such as ripples. However, the theory of gravity waves may yield waves of very short length which cannot be ignored and cause substantial difficulties in modelling them as described above. These singular and highly oscillatory properties being manifestly non-physical, it is expected that the surface tension plays an important role in modelling surface waves. In fact, the capillary-gravity waves have been studied since Kelvin (1871) as summarized in Wehausen & Laitone (1960). A classical analysis of asymptotic behavior of gravity waves including the effect of surface tension was given in Crapper (1964). The wave patterns of capillary-gravity waves in deep water were studied in Yih & Zhu (1989). Surface waves affected by surfactants (which induce a variation of surface tension) were recently studied by Zilman & Miloh (2001). More recently, Chen (2002) gave a updated analysis on the steady ship waves including the effect of surface tension. It is shown that the introduction of surface tension in the formulation of ship waves eliminates the singularity of ship waves in the region near the track of the source point at the free surface. Further to this study, we intend now to perform some introductory analysis on the transient waves due to an impulsive source, especially, at large time as well as in the region near the disturbance.

We define a reference system with the (x, y)-plane coincided with the undisturbed free surface and the z-axis oriented positively upward. In deep water, the solution to the initial-value and boundary-value problem satisfies the Laplace equation

$$\Delta G(P', t', P, t) = \delta(x-x')\delta(y-y')\delta(z-z')\delta(t-t') \qquad z \le 0\ ;\ |t| < \infty \tag{1a}$$

in which we denote $P'(x', y', z')$ and $P(x, y, z)$ as the source and field point respectively, the boundary condition over the free surface (Wehausen & Laitone, p636, eq.24.27)

$$[\partial_{tt} + g\partial_z + (T/\rho)\partial_{zzz}]G(P', t', P, t) = 0 \qquad z = 0\ ;\ t \ge t' \tag{1b}$$

the condition at infinity $\nabla G(P', t', P, t) \to 0$ for $(x-x')^2 + (y-y')^2 + (z-z')^2 \to \infty$, and the initial conditions $G(P', t', P, t) = 0 = \partial_t G(P', t', P, t)$ for $z = 0$ and $t \le t'$.

In above equations, g is the acceleration due to gravity, ρ water mass density and T surface tension on the air-water interface. The solution of the above problem can be written as :

$$G(P', t', P, t) = \delta(t-t')G^S(P', P) + H(t-t')G^F(P', t', P, t) \tag{2a}$$

with $G^S(P', P)$ the simple singularities (Rankine sources) given by

$$4\pi G^S(P', P) = -1/r + 1/r' \tag{2b}$$

and $G^F(P',t',P,t)$ taking account of free-surface effect. By using the Fourier-Hankel transform to the boundary condition on the free surface (1b) and the integral representation of the simple singularities $G^S(P',P)$, it is straightforward to obtain

$$2\pi G^F(P',t',P,t) = -\int_0^\infty e^{k(z+z')} J_0(kR)\sqrt{gk+(T/\rho)k^3}\sin\left[\sqrt{gk+(T/\rho)k^3}\,(t-t')\right] dk \tag{2c}$$

In (2b) and (2c), we have used

$$r = \sqrt{R^2+(z-z')^2}\ ; \quad r' = \sqrt{R^2+(z+z')^2} \quad \text{with} \quad R = \sqrt{(x-x')^2+(y-y')^2}$$

$H(\cdot)$ and $J_0(\cdot)$ are the Heaveside function and the zeroth-order Bessel function of the first kind, respectively. If we use L as a reference length to write the non-dimensional quantities as

$$\tau = (t-t')\sqrt{g/L}\ ; \quad (v,h) = (z+z',R)/L\ ; \quad \sigma = \sqrt{T/(\rho g L^2)}$$

the free-surface Green function is expressed

$$2\pi G^F(P',t',P,t) = -\sqrt{g/L^3}\,\phi(v,h,\tau) \tag{3a}$$

such that

$$\phi(v,h,\tau) = \int_0^\infty e^{kv}\omega J_0(kh)\,\sin(\omega\tau)\,dk \quad \text{with} \quad \omega(k) = \sqrt{k+\sigma^2k^3} \tag{3b}$$

The expression (3b) keeps the same form as that of pure-gravity waves. If we take $\sigma = 0$, the frequency $\omega(k,\sigma)$ is reduced to the pure-gravity $\omega^0(k) = \sqrt{k}$. In this classical case, Newman (1992) summarized his comprehensive analysis and the algorithms for numerical computations. At the limit $v \to 0$, it was given in Newman (1992, Eq.7.27)

$$\phi(v\to 0,h,\tau) \approx \tau/(h^2\sqrt{2})\sin[\tau^2/(4h)]\,e^{v\tau^2/(4h^2)} \quad \text{for} \quad \tau/\sqrt{h} \to \infty \tag{4}$$

which is associated with the contribution from the *unique* saddle point at $k = \tau^2/(4h^2)$ located at the real k-axis defined by the derivative of the phase function involved in the oscillatory part of the integrand in (3b)

$$\psi(k,a) = \omega - ka \tag{5}$$

with $a = h/\tau$ as the parameter. Unlike the pure-gravity case, we have *two* distinct saddle points k_g and k_T satisfying $\psi' = 0$ for $a \gg \sqrt{\sigma}$

$$k_g = \frac{1}{4a^2}\left[1+\frac{5}{2^4}\left(\frac{\sigma}{a^2}\right)^2+\frac{54}{2^8}\left(\frac{\sigma}{a^2}\right)^4+\frac{741}{2^{12}}\left(\frac{\sigma}{a^2}\right)^6+\cdots\right] \tag{6a}$$

$$k_T = \frac{4a^2}{9\sigma^2}\left[1+\frac{3^3}{2^4}\left(\frac{\sigma}{a^2}\right)^2-\frac{5\cdot 3^6}{2^8}\left(\frac{\sigma}{a^2}\right)^4+\frac{30\cdot 3^9}{2^{12}}\left(\frac{\sigma}{a^2}\right)^6+\cdots\right] \tag{6b}$$

When a is of the same order as $\sqrt{\sigma}$, the wavenumbers k_g and k_T become close and in particular,

$$a = h/\tau = a_0 = \left(3^{3/8}(\sqrt{3}-1)/\sqrt{2\sqrt{2-\sqrt{3}}}\right)\sqrt{\sigma} \approx 1.0862595\sqrt{\sigma}$$

we have

$$k_g = k_T = k_0 = \left(\sqrt{(2-\sqrt{3})/\sqrt{3}}\right)/\sigma \approx 0.3933199/\sigma \tag{7a}$$

and $\varphi'' = \omega_0'' = 0$ as two saddles points coalesce, while the third-order derivative $\varphi''' = \omega_0''' = 4\sigma^2k_0^2/\omega_0^5 > 0$. Finally, for a value of $a > a_0$ close to a_0, we write

$$a = a_0\sqrt{1+s^2} \quad \text{then} \quad s = \sqrt{(a/a_0)^2-1} \tag{8a}$$

here s is the parameter and the saddle points are given

$$k_g = k_0(1-a_1s+a_2s^2-a_3s^3+a_4s^4-a_5s^5+\cdots) \tag{8b}$$

$$k_T = k_0(1+a_1s+a_2s^2+a_3s^3+a_4s^4+a_5s^5+\cdots) \tag{8c}$$

with

$$
\begin{aligned}
a_1 &= \sqrt{1+1/\sqrt{3}} & a_5 &= (180\sqrt{3}-183)a_1/3456 \\
a_2 &= 2/3 & a_6 &= (13+10\sqrt{3})/648 \\
a_3 &= (21-10\sqrt{3})a_1/72 & a_7 &= (2097-2366\sqrt{3})a_1/82944 \\
a_4 &= -(3+2\sqrt{3})/108 & & \cdots
\end{aligned}
$$

When $a = h/\tau < a_0$, there exist two complex saddle points

$$k_{g,T} = k^R \mp ik^I \tag{9a}$$

with (k^R, k^I) given by using $s = \sqrt{1-(a/a_0)^2}$ as the parameter

$$k^R = k_0(\,1 \;\; - a_2 s^2 + a_4 s^4 - a_6 s^6 + \cdots) \tag{9b}$$

$$k^I = k_0(a_1 s - a_3 s^3 + a_5 s^5 - a_7 s^7 + \cdots) \tag{9c}$$

For $a \ll a_0$, we write $s = a/\sqrt{\sigma}$ and have

$$
\begin{aligned}
\sigma k^R &= \frac{s}{3^{5/4}} + \frac{s^2}{9} - \frac{s^3 3^{1/4}}{54} - \frac{s^7 3^{1/4} 65}{11664} + O(s^9) \\
\sigma k^I &= \frac{1}{3^{1/2}} - \frac{s}{3^{5/4}} - \frac{s^3 3^{1/4}}{54} - \frac{s^4 5}{3^{1/2} 54} - \frac{s^7 3^{1/4} 65}{11664} - \frac{s^8 35}{3^{1/2} 1944} + O(s^9)
\end{aligned}
$$

so that at $a = 0$, the saddle points are located at the imaginary k-axis since

$$k_{g,T} = \mp i/(\sqrt{3}\sigma) \tag{9d}$$

By using the method of steepest descent, we obtain the asymptotic behaviors of $\phi(v \to 0, h, \tau)$ at above three different cases with respect to the parameter a :

$$\phi(v \to 0, h, \tau) \approx \left[\sqrt{\frac{1+\sigma^2 k_g^2}{-\omega_g''}} \sin(\tau\omega_g - k_g h)\, e^{k_g v} + \sqrt{\frac{1+\sigma^2 k_T^2}{\omega_T''}} \cos(\tau\omega_T - k_T h)\, e^{k_T v} \right] / \sqrt{h\tau} \tag{10a}$$

with $\omega_{g,T} = \omega(k_{g,T})$ for $a \gg a_0$.

$$\phi(v \to 0, h, \tau) \approx \sin(\tau\omega_0 - k_0 h + \pi/4)\, \frac{e^{k_0 v}\sqrt{2\pi(1+\sigma^2 k_0^2)}}{(\tau\omega_0'''/2)^{1/3}\sqrt{a\tau}}\, \mathrm{Ai}\!\left[\frac{(\omega_0' - a)\tau^{2/3}}{(\omega_0'''/2)^{1/3}}\right] \tag{10b}$$

for $a \approx a_0$ in which $\mathrm{Ai}(\cdot)$ is the Airy function defined in Abramowitz & Stegun (1967) and $\{\omega_0, \omega_0', \omega_0'', \omega_0'''\} = \{\omega(k_0), \omega'(k_0), \omega''(k_0), \omega'''(k_0)\}$, and finally for $a \ll a_0$

$$\phi(0, h, \tau) \approx -4/\tau^3 + 5!(3h^2 + 4\sigma)/\tau^7 + O(\tau^{-11}) \tag{10c}$$

which is associated with the contribution from the end point of (3b), i.e. at the origin $k = 0$ since that from the complex saddle points is exponentially small.

The above equation (10a) indicates that the amplitude of $\phi(0, h, \tau)$ is of order $O(1/\tau^{1/2})$ as $\tau \to \infty$ for $h/\tau = a > a_0$. If the ratio $a = h/\tau$ keeps constant, the amplitude of $\phi(0, h, \tau)$ is of order $O(1/\tau)$ for $\tau \to \infty$. There are two wave systems associated with k_g and k_T corresponding to the first and second terms in (10a), respectively. At $\sigma \to 0$, the wave system associated with k_T - the second term - disappears while the first term associated with k_g is reduced to the same as (4) for the pure-gravity waves. The k_g- and k_T-waves are thus called as the gravity-dominant and capillary-dominant wave systems, respectively. The two wave systems are merged into one as expressed by (10b) at $a = a_0$. For $a < a_0$, the amplitude of wave systems decreases exponentially and the leading term is no-wavy and decreases at the time rate $-4/\tau^3$ everywhere as indicated by the first term in (10c). Physically, $a = h/\tau$ is the propagation velocity of capillary-gravity waves which has a minimum limit a_0. In other words, we should find a calm region near the disturbance whose radius $h_0 = a_0\tau$ increases with time. To confirm above analyzes, we have performed the numerical integration of (3b) in the complex k-plane and results are presented in the following figures.

The gravity-dominant waves, capillary-dominant waves and their sum are shown on Fig.1 respectively by the thick solid, thin solid and dashed lines for $(v, h) = (-0.01, 0.5)$ and τ varying from 0 to 12. The two wave systems present for $\tau < 8.837$ are dominated by the gravity-dominant part, decreases rapidly for

Fig.1: Gravity-dominant, capillary-dominant and the total capillary-gravity waves

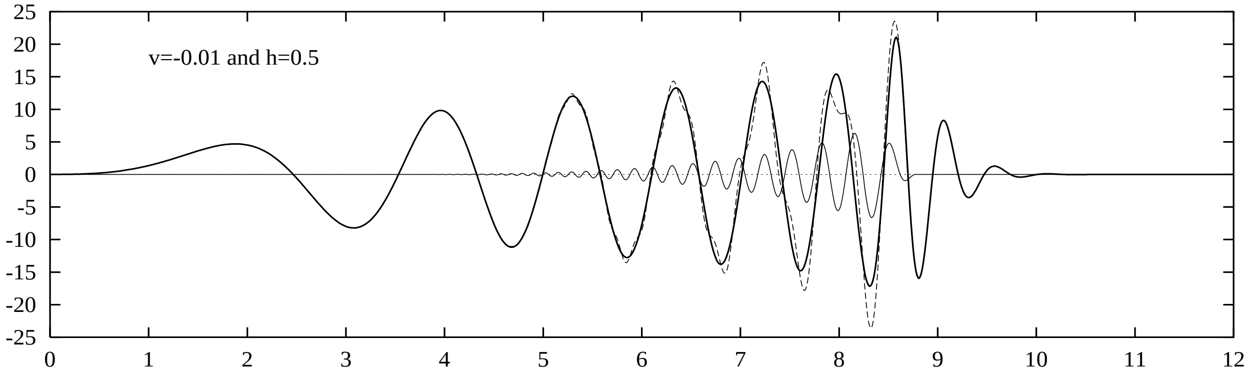
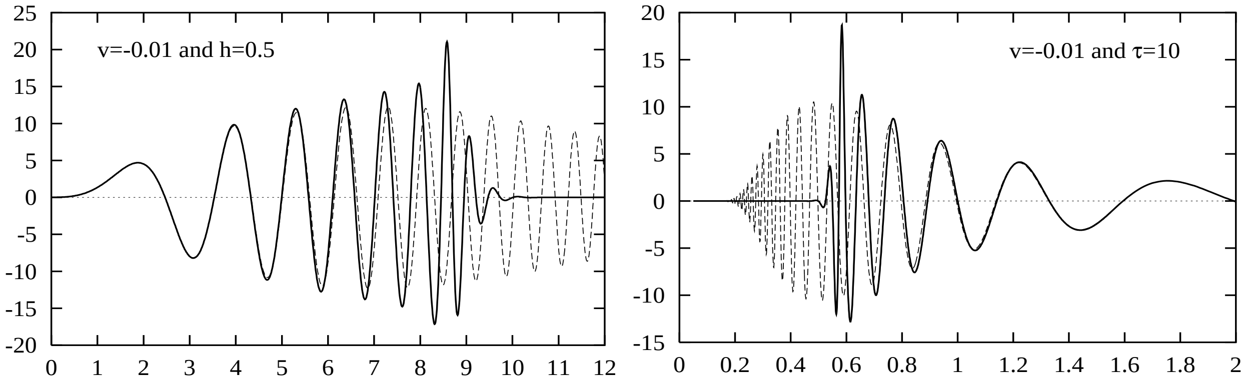


Fig.2a (left) & **Fig.2b** (right): Gravity-dominant waves (solid line) and pure-gravity waves (dashed line)



$8.837 < \tau < 10$ and disappear completely for $\tau > 10$, while the pure-gravity waves persist as shown on Fig.2a (left column) by the dashed line and compared with the gravity-dominant waves reproduced by the solid line. On Fig.2b (right column), the comparison of gravity-dominant waves (solid line) and pure-gravity waves (dashed line) is presented for $(v, \tau) = (-0.01, 10)$ and h varying from 0 to 2. The capillary-gravity waves illustrated on Fig.2a starts with the same value as pure-gravity waves but the difference between them in both magnitude and wavelength increases with time. At large time (Fig.2a) as well as in the region near the disturbance (Fig.2b), the capillary-gravity waves disappear while the small pure-gravity waves continue their manifestation. These properties of capillary-gravity waves are welcome and believed to be much profitable in the numerical solution to floating body problems, especially, in the evaluation of the waterline and time-convolution integrals.

We acknowledge Dr. Alain H. Clément of École Centrale de Nantes for kindly providing the results of pure-gravity waves received on the first day of Year 2003 and presented in Fig.2a & Fig.2b for comparison.

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Question by : H. Bingham
It is easy to show that the effect of surface tension on dispersion is less than 10% for waves O(1cm) long, yet you seem to find a large effect over a large range of wavenumbers. Can you comment on and explain this ?

Author's reply:
The surface tension is dominant effect over gravity for short waves as we may check from the dispersion relation. The recent work shows that the effect of surface tension can even be important for waves of length up to of an order O(m). The primary objective of the present work is to show the deficiency of pure-gravity theory in describing free-surface waves which is the origin of numerical troubles in modelling potential flow around a ship, and to get remedy from introducing the surface tension.
At this stage, it's not yet clear for us to estimate the final importance of surface tension in modelling surface waves from a practical purpose of view.
