Water wave diffraction by vertical circular cylinder in partially frozen sea

Malenica Š.[1] **& Korobkin A.A.**[2]

(1) BUREAU VERITAS - DR, Paris, France (sime.malenica@bureauveritas.com)
(2) Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia (kaaa@hydro.nsc.ru)

Introduction

The linear diffraction of an incident monochromatic wave by the vertical circular cylinder in the sea with partially frozen free surface, is considered. It is assumed that the ice sheet is circular and fixed to the cylinder (deflection of the connecting point as well as the slope are zero) while the ice sheet is free at its end (shear force and moment are zero). Potential flow is assumed for the fluid and thin plate theory for the ice deflections. Furthermore, the ice sheet is assumed to have zero thickness and to be homogeneous with constant density and constant flexural rigidity. The basic configuration is shown on figure 1.

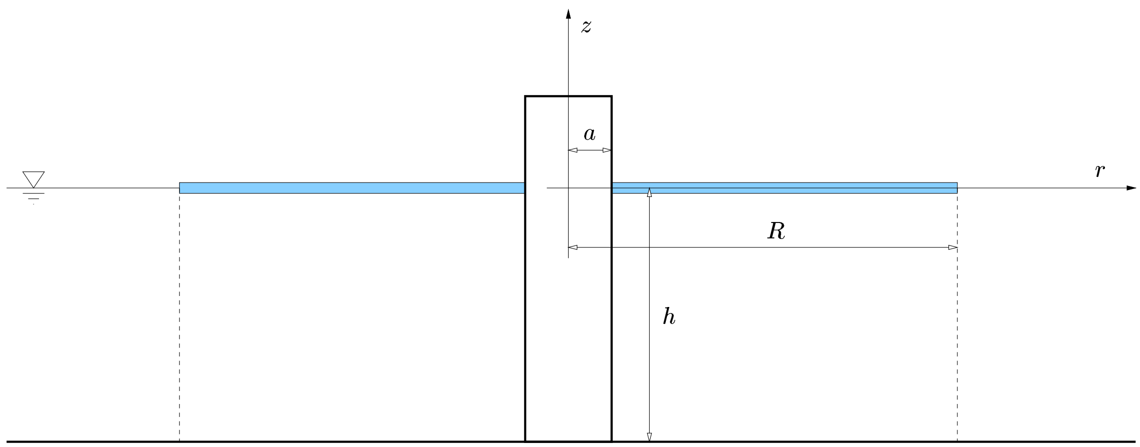

Figure 1: *Basic configuration and definitions.*

Mathematical model

First of all, we divide the fluid domain in two parts: the inner part below the ice sheet ($a < r < R$), and the outer part without ice $r > R$. Under the potential flow assumptions, we end up with the boundary value problem (BVP) for velocity potential $\Phi(x,t)$ in whole fluid domain. This potential should satisfy Laplace equation in the fluid domain ($\Delta\Phi = 0$), no flow condition on the cylinder and on the sea bottom ($\partial\Phi/\partial n = 0$), classical linear free surface condition on the outer free surface ($\partial^2\Phi/\partial t^2 + g\partial\Phi/\partial z = 0$, $z = 0, r > R$) and the coupling boundary condition on the inner free surface ($z = 0, a < r < R$), which we discuss below.

Boundary condition on the ice-water interface

In order to obtain the boundary condition on the interface between the ice sheet and the water, we should couple the fluid flow with the ice sheet deflections. To do this, we need to consider the kinematic (equality of normal velocities) and dynamic (equality of the pressures) conditions. We start by writing the governing equation for thin plate deflections:

$$M\frac{\partial^2 W}{\partial t^2} + D\Delta_0^2 W = P \tag{1}$$

where $W(x,t)$ is the deflection, M is the mass of unit area, D is flexural rigidity, P is the external pressure and Δ_0 denotes horizontal Laplace operator ($\Delta_0 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$). The dynamic condition states that the pressure is equal to the hydrodynamic pressure which can be calculated from Bernoulli equation:

$$P = -\varrho g W - \varrho \frac{\partial \Phi}{\partial t} \tag{2}$$

On the other hand, the kinematic condition requires the equality of the normal velocities:

$$\frac{\partial W}{\partial t} = \frac{\partial \Phi}{\partial z} \tag{3}$$

After taking the time derivative of (1) we can combine two equations in one:

$$(M\frac{\partial^2}{\partial t^2} + D\Delta_0^2 + \varrho g)\frac{\partial \Phi}{\partial z} + \varrho \frac{\partial^2 \Phi}{\partial t^2} = 0 \tag{4}$$

We assume now the periodicity of the process $[\Phi(x,t) = \Re\{\varphi(x)e^{-i\omega t}\}, W(x,t) = \Re\{w(x)e^{-i\omega t}\}]$ and we write the frequency domain equivalent of the above boundary condition:

$$(-\omega^2 M + D\Delta_0^2 + \varrho g)\frac{\partial \varphi}{\partial z} - \varrho \omega^2 \varphi = 0 \tag{5}$$

In order to close the mathematical model we should also define the boundary conditions at the edges of the ice sheet. As stated in the introduction, at the connection point (circle) with the vertical cylinder we require that the deflection and the slope are zero:

$$w(a) = 0 \quad , \quad \frac{\partial w}{\partial r}(a) = 0 \tag{6}$$

while the shear force and bending moment are zero at ice sheet end ($r = R$):

$$\Delta_0 w - (1-\nu)\Big[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\Big] = 0 \quad , \quad \frac{\partial \Delta_0 w}{\partial r} + \frac{1-\nu}{r}\frac{\partial}{\partial r}\Big[\frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\Big] = 0 \tag{7}$$

where ν is Poisson's ratio.

Solution methodology

In order to solve the above defined BVP, we propose to use the method of matched eigenfunction expansions. Due to the axisymetric geometry of the domain, we first expand all quantities in the Fourier series in circumferential direction:

$$\varphi(r,z,\theta) = \sum_{m=0}^{\infty} \epsilon_m \varphi_m(r,z)\cos m\theta \quad , \quad w(r,\theta) = \sum_{m=0}^{\infty} \epsilon_m w_m(r)\cos m\theta \quad , \quad ... \tag{8}$$

where ϵ_m is equal to 1 for $m = 0$ and 2 for $m > 0$.
For each domain, we consider now the eigenfunction expansions for potential φ_m in radial and vertical directions.

Eigenfunction expansion in the outer domain $r > R$

Let first note that in the outer region the total potential is divided into the incident and diffracted parts:

$$\varphi^{out} = \varphi_I + \varphi_D^{out} \tag{9}$$

Since the BVP for the outer domain is the classical BVP for water wave diffraction and, due to the relatively simple free surface condition ($-\omega^2\varphi^{out} + g\partial\varphi^{out}/\partial z = 0$), the eigenfunction expansion can be easily found in the form:

$$\varphi_{Dm}^{out}(r,z) = \sum_{n=0}^{\infty} A_{mn} f_n(z) H_m(k_n r) \qquad , \qquad f_n(z) = \frac{\cosh k_n(z+h)}{\cosh k_n h} \tag{10}$$

where the eigenvalues k_n are the roots of the dispersion relation $\omega^2/g = k \tanh kh$.

Note that the solution of the dispersion relation gives one real root k_0 and infinite number of imaginary roots $k_n,\ n = 1, \infty$. This imply that the eigenfunctions $f_n(z)$ becomes cosine functions and Hankel functions H_m becomes the modified Bessel functions K_m, for $n > 0$. Note also that the set of the eigenfunctions $f_n(z),\ n = 0, \infty$ is orthogonal and complete.

For the incident potential we choose the classical regular sinusoidal wave of unit amplitude:

$$\varphi_I \quad = \quad -\frac{ig}{\omega} f_0(z) e^{ik_0 r \cos\theta} = -\frac{ig}{\omega} f_0(z) \sum_{m=0}^{\infty} \epsilon_m i^m J_m(k_0 r) \cos m\theta \tag{11}$$

Eigenfunction expansion in the inner domain $a < r < R$

Due to the same governing equation in the fluid (Laplace equation) the eigenfunctions in vertical direction are written in the form similar to (10):

$$F_n(z) = \frac{\cosh \mu_n (z + h)}{\cosh \mu_n h} \tag{12}$$

However, due to the higher order terms in the free surface condition (5), the situation is more complicated, and the eigenvalues μ_n, are the roots of the following dispersion relation:

$$\frac{\omega^2}{g} = (1 - \frac{\omega^2 M}{\varrho g} + \frac{D}{\varrho g}\mu^4)\mu \tanh \mu h \tag{13}$$

In contrast to the outer domain, the solution of this equation consists of one real root μ_0, infinite number of imaginary roots $\mu_n,\ n = 1, \infty$ and two complex roots μ_{-1}, μ_{-2}. Also it can be shown that two complex roots are related to each other by the relation $\mu_{-2} = -\mu^*_{-1}$, with asterix denoting the complex conjugate. Since the set of eigenfunctions $F_n(z),\ n = 0, \infty$ represents already complete and orthogonal set of functions, the two remaining functions can be expressed as a linear combination of these functions. We can write:

$$F_{-1}(z) = \sum_{k=0}^{\infty} \alpha_k F_k(z) \quad , \quad F_{-2}(z) = \sum_{k=0}^{\infty} \alpha^*_k F_k(z) \tag{14}$$

where the coefficients α_k can easily be found using the orthogonal property of the basis functions $F_k(z),\ k = 0, \infty$,:

$$\alpha_k = \frac{\int_{-h}^{0} F_{-1}(z) F_k(z) dz}{\int_{-h}^{0} F_k^2(z) dz} \tag{15}$$

The general solution in the inner region can now be written in the following form:

$$\varphi_m^{in}(r, z) = \sum_{n=-2}^{\infty} F_n(z)[B_{mn} J_m(\mu_n r) + C_{mn} Y_m(\mu_n r)] \tag{16}$$

Linear system of equations for unknown coefficients

In order to solve for the unknown potentials, we have to truncate the infinite series in the expressions (10,16) and derive the linear set of equations for the unknown coefficients A_{mn}, B_{mn}, C_{mn}. Let assume that in the outer expansion (10) the series is truncated after L elements and in the inner expansion (16) after N elements. This means that, in order to properly close the problem we need $(L+1)+2\times(N+1)+4$ equations. These equations will be obtained after taking into account the boundary condition on the cylinder, matching conditions on the interface between inner and outer domain and the end conditions for the ice sheet. It is important to note that each Fourier mode have to be considered separately.

Boundary condition on the cylinder

As stated before, the boundary condition on the cylinder is the classical no flow condition requiring that the radial derivative of potential is zero:

$$\frac{\partial \varphi_m^{in}}{\partial r}(a) = 0 \tag{17}$$

Since there is only $L + 1$ independent eigenfunctions in the inner region, this condition will give $L + 1$ equations. The coefficients of these equations are obtained by using the orthogonality property of the eigenfunctions.

Matching of inner and outer eigenfunctions expansions

The matching conditions require the continuity of the pressure and the velocity through the matching region:

$$\left\{\varphi^{in} = \varphi_I + \varphi_D^{out}\right\}_{r=R} \quad , \quad \left\{\frac{\partial \varphi^{in}}{\partial r} = \frac{\partial}{\partial r}(\varphi_I + \varphi_D^{out})\right\}_{r=R} \tag{18}$$

If we multiply the first equation by the functions $F_n(z),\ n = 0, L$, and integrate from $-h$ to 0 both sides of equation we obtain additional $L+1$ equations. By repeating the same procedure with the second condition, but using the functions $f_n(z),\ n = 0, N$, the new $N+1$ equations are obtained.
Note that the choice of the eigenfunctions for each matching condition is arbitrary.

Boundary conditions for the ice sheet ends

As we can see 4 equations are still missing. They will be obtained by considering the boundary conditions for the ice sheet ends (6,7). To do this let write the expression for the plate deflections (3) in convenient form:

$$w(r,\theta) = i\omega \frac{\partial \varphi^{in}}{\partial z}(r,\theta,0) = \frac{i}{\omega} \sum_{m=0}^{\infty} \sum_{n=-2}^{\infty} \epsilon_m S_n [B_{mn} J_m(\mu_n r) + C_{mn} Y_m(\mu_n r)] \cos m\theta \tag{19}$$

where : $S_n = \varrho\omega^2/[\varrho g - \omega^2 M + D\mu_n^4]$.
The first two conditions (6) at the connecting circle on the cylinder give following two equations:

$$\sum_{n=-2}^{L} S_n [B_{mn} J_m(\mu_n a) + C_{mn} Y_m(\mu_n a)] = 0 \tag{20}$$

$$\sum_{n=-2}^{L} \mu_n S_n [B_{mn} J_m^{'}(\mu_n a) + C_{mn} Y_m^{'}(\mu_n a)] = 0 \tag{21}$$

while the free boundary conditions at the outer circle (7) give:

$$\sum_{n=-2}^{L} \mu_n^2 S_n \Big\{ \Big[1 - \frac{(1-\nu)m^2}{(\mu_n R)^2}\Big] \Big[B_{mn} J_m(\mu_n R) + C_{mn} Y_m(\mu_n R)\Big] + \frac{1-\nu}{\mu_n R}\Big[B_{mn} J_m^{'}(\mu_n R) + C_{mn} Y_m^{'}(\mu_n R)\Big]\Big\} = 0 \tag{22}$$

$$\sum_{n=-2}^{L} \mu_n^3 S_n \Big\{ \frac{(1-\nu)m^2}{(\mu_n R)^3} \Big[B_{mn} J_m(\mu_n R) + C_{mn} Y_m(\mu_n R)\Big] - \Big[1 + \frac{(1-\nu)m^2}{(\mu_n R)^2}\Big]\Big[B_{mn} J_m^{'}(\mu_n R) + C_{mn} Y_m^{'}(\mu_n R)\Big]\Big\} = 0 \tag{23}$$

These equations complete the linear system, solution of which provides the coefficients A_{mn}, B_{mn}, C_{mn}. The system is solved numerically by the classical methods.
The results for both horizontal and vertical force components on the cylinder and bending stresses in the plate will be presented at the Workshop.

References

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Question by : M. Kashiwagi
In the case of a bottom mounted vertical cylinder the evanescent waves will be zero in the conventional gravity waves. However, in the present problem the inner-domain solution includes various eigenfunctions. Why are those eigenfunctions needed in the present case?

Author's reply:
The evanescent modes are present in the inner solution because the free surface condition change in this region. This means that the vertical eigenfunctions in the inner and outher domain will not be the same. The consequence is that the boundary conditions, on the cylinder and at the intersection radius, can not be satisfied without including the evanescent modes.

Question by : R. Porter
In your abstract you claim that the eigenfunctions Fn(z) in the inner region form a complete and orthogonal set for n=0,1, …, and that F-1(z) and F-2(z) can be expanded as a linear combination of Fn(z), n=0,1,… Is this correct? In the abstract of Evans & Porter, for example, we show that Fn(z) are non-orthogonal. I believe that the proof of completeness is not trivial.

Author's reply:
It was a missprint in the abstract. The functions fn(z) are orthogonal but not Fn(z). In order to express the F-1(z) and F-2(z) by Fn(z), n=0,1,2,…, we have to solve a linear system of equations in order to found the required representation. As for the completeness, we agree that the proof is non-trivial, but we can say that in the present case the solution converges very quickly.