How special are trapping structures?

P. McIver
Department of Mathematical Sciences, Loughborough University, UK. Email: p.mciver@lboro.ac.uk

SUMMARY
Some structures floating in water can support a trapped mode which is a free oscillation of the fluid with finite energy. In the two-dimensional water-wave problem the best known trapping structure has two fixed surface-piercing elements and the fluid motion is essentially confined to the region between the structures. In this paper, the perturbation of such structures is examined to assess the importance of the particular geometries. Two types of perturbation are examined, namely changes in the spacing between the structural elements and changes in the geometry of individual elements.

1 INTRODUCTION

Trapped modes are free oscillations with finite energy of an unbounded fluid for which the fluid motion is essentially confined to the vicinity of a fixed structure, and thus a trapped mode does not radiate energy to infinity. For structures in the open sea, such trapped modes are possible only for particular shapes of structure, and then they occur only for discrete frequencies of oscillation of the fluid.

The existence of a trapped mode for a structure has notable consequences in both the frequency and time domains. For example, in the frequency domain there is singular behaviour in the added mass at the trapped-mode frequency [1], while in the time domain a forced motion of a trapping structure usually results in a persistent fluid oscillation at the trapped-mode frequency [2].

Here attention is restricted to two-dimensional problems and Cartesian coordinates (x, z) are chosen with z directed vertically upwards and with origin in the mean free surface. The fluid domain is bounded below by a flat rigid bed at $z = -h$ but extends to infinity in both the positive and negative x directions. A typical surface-piercing trapping structure for the two-dimensional problem, constructed by the method described in [3], is shown in figure 1 for the case of deep water ($h \to \infty$). The flow field for this particular structures is generated by two wave sources placed in the free surface at $x = \pm\pi/2$ (marked by filled circles in the figure), and there are no waves at infinity provided $K = \omega^2/g = 1$, where ω is the frequency of the fluid oscillations and g is the acceleration due to gravity. The structure is formed from two streamlines of the flow.

A trapped mode for a fixed structure such as that in figure 1 consists of a fluid motion that is essentially confined to the region between the structural elements with decay of the fluid motion as $|x| \to \infty$. In this example, the fluid motion is symmetric about $x = 0$ and the fluid motion between the elements is a "pumping" motion without any free-surface nodes. Structures have been constructed that support trapped modes that are both symmetric and anti-symmetric about $x = 0$ and with any number of nodes on the section of free surface between the elements. In the example illustrated in figure 1 there is symmetry of the geometry about $x = 0$, but this need not be the case.

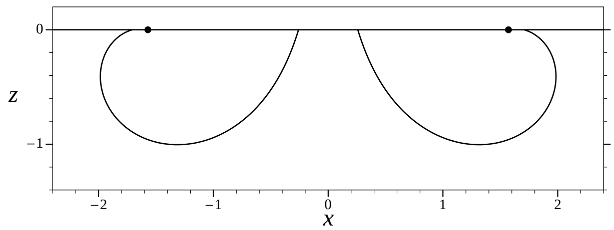

Figure 1: Example of a trapping structure.

In general, an arbitrarily small change in the geometry of a trapping structure will give a structure that does not support trapped modes (although, for the structures here, there are certain perturbations that preserve the trapping property). The aim is to examine how "special" trapping structures are. How are the frequency- and time-domain properties of a trapping structure affected by perturbations in geometry that destroy the trapping property? Here attention will be restricted to two situations. In section 2 the affects on the added mass of changes in the spacing between structural elements will be examined. In section 3 the affects on the time-domain solution of changes in the geometry of the structural elements will be examined.

2 CHANGES IN ELEMENT SPACING

Here the structures considered are perturbations of that in figure 1. The heave added mass μ for the trapping structure itself is shown in figure 2. Rapid changes in μ are associated with the singularities of the frequency-domain potential $\phi(x, z, \omega)\,\mathrm{e}^{-\mathrm{i}\omega t}$ in the complex ω plane (here t is time). For a trapped mode there is a pole of ϕ on the real ω axis and as the geometry is perturbed away from a trapping structure this pole moves into the lower half of the complex ω plane. As a function of real frequency, the added mass undergoes very rapid changes near a trapped,

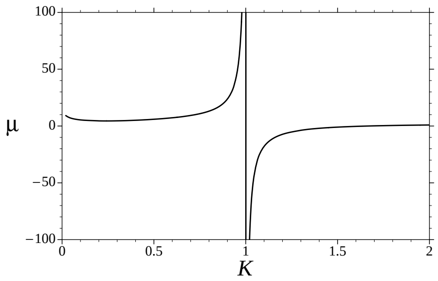

Figure 2: Heave added mass μ vs. frequency parameter K for the trapping structure shown in figure 1.

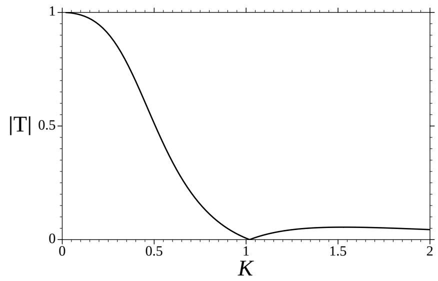

Figure 3: Modulus of transmission coefficient T vs. frequency parameter K for one element of trapping structure.

or near-trapped, mode for which there is a pole in ϕ respectively on or close to the real ω axis.

The computations were made with a panel code and because of numerical error the "singular" behaviour of the added mass does not occur exactly at $K = 1$ (in fact it occurs at $K = 1.0003$). For a trapping structure the added mass becomes unbounded as the trapped-mode frequency is approached but, again due to numerical error, in the computations the extremes of the added mass are finite (but in excess of 10^5 in this example). The value of K corresponding to a sign change in the added mass will be denoted by K_0, and the positive and negative extreme values of the added mass by μ_+ and $-\mu_-$ respectively (it can be shown that $\mu_+ \approx \mu_-$ for a near-trapped mode).

If the structural elements are moved apart a distance d, say ($d = 0$ corresponds to the trapping structure), then K_0 and $\mu_\pm$ all decrease until the characteristic shape of figure 2 is no longer discernible. Simultaneously, a similar feature in the added mass grows and moves toward $K = 1$ from higher frequencies. A small sample of results is given in table 1. For each value of d, data is given for the "singular" feature in the added mass of lowest frequency.

d	K_0	μ_+	μ_-
0	1.000	$> 10^5$	$> 10^5$
0.050	0.911	1.2×10^4	1.2×10^4
0.500	0.585	17.2	18.6
1.000	0.452	5.1	5.8
2.000	1.484	3.1×10^2	3.5×10^2
3.085	1.019	$> 10^5$	$> 10^5$

Table 1: Changes in the singular behaviour of the added mass as the element spacing d is increased.

The appearance of a second "singular" feature in the data of table 1 can be explained in terms of a wide-spacing approximation based on the properties of a single structural element. Provided that the two elements of a structure are sufficiently widely spaced for local fields to be negligible in hydrodynamic interactions, the following heuristic argument predicts the existence of trapped modes [4]. If an element totally reflects a plane wave (so that the transmission coefficient $T = 0$) at a particular frequency, then by appropriate choice of the spacing between the element and a second element formed by the mirror image of the first in $x = 0$, it is possible to construct a standing wave oscillation between the elements and have the fluid motion decay to zero as $|x| \to \infty$ (the propagating wave components of the standing wave are both totally reflected). In figure 3 $|T|$ for one element of the structure shown in figure 1 is given as a function of frequency and, as can be seen, there is a minimum in $|T|$ close to zero near $K = 1$ (in fact it occurs at $K = 1.019$). Under the assumption that this minimum is indeed a zero, the wide-spacing argument predicts trapped modes for $K = 1.019$ at an infinite sequence values of the spacing parameter d some of which are given in table 2. Two types of mode are predicted, namely modes that are either symmetric or antisymmetric about $x = 0$. In all cases near-singular behaviour in the heave added mass is found and the existence of the symmetric modes explains the results of table 1 in which $\mu_\pm$ increase as d approaches 3.085 (on the basis of numerical calculations, it can not be determined whether or not a true trapping structure can be obtained in this way). The antisymmetric modes are associated with near-singular behaviour in the surge added mass.

symmetric	antisymmetric
0.002	1.544
3.085	4.627
6.168	7.710

Table 2: Values of the spacing d for which trapped modes are predicted by the wide-spacing approximation.

3 CHANGES IN ELEMENT GEOMETRY

Attention is now turned to changes in the elements of the structure and the associated effects in the time domain. Again the structure is constructed from wave sources, but here the structure is in fluid of finite depth h and, for this example, the trapped mode has an oscillation frequency $\omega_0 = \sqrt{(4 \tanh 4)g/h} \approx 1.99933\sqrt{g/h}$. The corresponding fluid oscillation is symmetric about $x = 0$ and between the structural elements there are two free-surface nodes. Outside the elements, the fluid motion decays to zero as $|x| \to \infty$. Note that the construction guarantees the existence of only a single trapped mode. Calculations are also presented for two other structures, a pair of half-immersed circles and a pair of half-immersed ellipses, with boundaries that intersect the free surface $z = 0$ at the same points as the trapping structure (see figure 4).

Time-domain calculations for the linear water-wave problem were made using the method described in [2]. The structure is moved vertically with the displacement

$$S(t) = \begin{cases} 0, & t < 0, \\ \alpha h(t/T)^3 \, \mathrm{e}^{-t/T}, & t \geq 0, \end{cases} \tag{1}$$

so that it is displaced and then brought back to rest; the constant $\alpha \ll 1$ ensures that linear theory applies, and the time scale $T = \sqrt{h/g}$.

The results in figure 5 are for the trapping structure and the two semicircles and show the free-surface elevation $\eta(0, t)$ at the mid point $x = 0$ between the two structural elements. For times $t < 30T$, $\eta(0, t)$ is noticeably different for the two structures, but for larger times the free-surface motions are very similar. The discrete Fourier transform was used to reveal the frequency content of the time signals; results are described in terms of the non-dimensional frequency $\Omega = \omega\sqrt{h/g}$, where ω is the transform variable. The fluid response is expected to display an oscillation in $\eta(0, t)$ at non-dimensional frequency $\Omega \approx 2.0$ (the trapped-mode frequency), but there should be no radiation to infinity at this frequency.

For both geometries of figure 5 there are three main peaks in the Fourier transform: a broad peak at $\Omega \approx 0.6$, and narrow peaks at $\Omega \approx 2.0$, and 2.8. When the time interval $0 < t/T < 50$ is excluded there are only two significant peaks at $\Omega \approx 2.0$ and 2.8 and a virtually zero transform in the low-frequency range. The combined oscillations at $\Omega \approx 2.0$ and 2.8 are observable for larger times in figure 5; they correspond to standing waves between the structural elements whose wavelength is approximately equal to, and one half of, the inter-element spacing, respectively. The low-frequency oscillation at $\Omega \approx 0.6$ is a pumping motion that dies out rapidly due to wave radiation and can be seen at smaller times in figure 5. The similarity between, and the positions of, the 'inner' parts of the two geometries near the free surface ensure that the high-frequency components of the fluid motions are also similar. On the other hand, the geometrical differences at depth and on the 'outer' parts of the structures significantly influence the low-frequency components of the motions.

Wave radiation is revealed by the Fourier transform of a far-field time signal corresponding to the results in figure 5. With the earlier times removed, this shows an isolated peak at $\Omega \approx 2.0$ for the half-immersed circles but no other significant peak for either structure. The absence of a peak at $\Omega \approx 2.0$ for the trapping structure is to be expected as, by construction, the trapped mode does not radiate energy at this frequency. No measurable radiation at $\Omega \approx 2.8$ could indicate that both structures possess a trapped mode at this frequency. However, it seems more likely that, because of the deep drafts relative to the wavelength, radiation is simply very small.

Time-domain results for the half-immersed ellipses are presented in figure 6. Here $\eta(0, t)$ is clearly different from either signal shown in figure 6. The Fourier transform of the complete signal shows a broad-band peak at $\Omega \approx 0.75$ and narrower peaks at $\Omega \approx 2.1$ and 2.9 (the three peaks are interpreted as above), although there is significant energy for almost all frequencies up to about $\Omega \approx 3$. The Fourier transform of the corresponding time signal for the far field shows a narrow peak at $\Omega \approx 2.1$ and significant components at almost all frequencies up to $\Omega \approx 3$. The shallow draft of this structure means that very narrow-banded near-resonances no longer occur within the frequency range studied, and hence the free-surface motion is more complex than those shown in figure 5 and has more frequency components of significant amplitude.

For a structure that supports trapped or near-trapped modes, the large-time asymptotics of the time-domain potential is related to the pole structure of the frequency-domain potential ϕ. To illustrate this the heave added mass for the three structures shown in figure 4 is given as a function of Ω in figure 7, and the rapid variations arising from the poles are readily seen. The main features of the time-domain calculations are all reflected in the added-mass curves shown in figure 7. For both the trapping structure and the half-immersed circles, there is a broad-banded local minimum at $\Omega \approx 0.6$ that corresponds to the initial pumping oscillation in the time domain, and for the persistent oscillations at $\Omega \approx 2.0$ and 2.8 there are corresponding narrow-banded changes in the added mass coefficient. On the other hand the curve for the half-immersed ellipses shows more broad-banded, less pronounced, peaks and a shift towards higher frequencies of the local extrema.

4 CONCLUSION

Geometrical perturbations of a trapping structure that appear to be significant can lead to structures whose properties in the frequency- and time-domains are difficult to distinguish from those of the original trapping structure.

5 ACKNOWLEDGEMENT

This work is supported by the UK Engineering and Physical Sciences Research Council under grant GR/M30937.

6 REFERENCES

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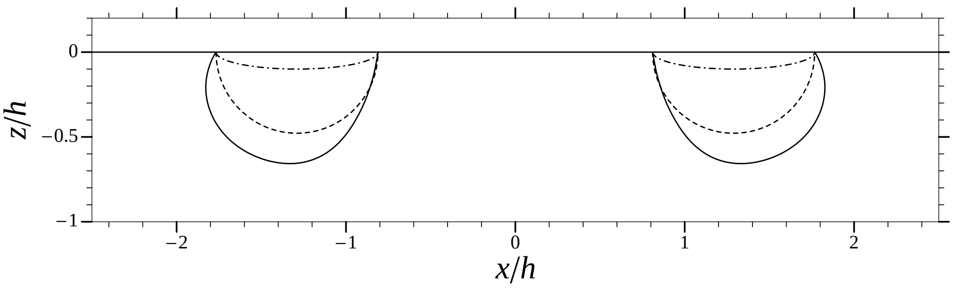

Figure 4: Geometries: trapping structure (———), pair of half-immersed circles (– – – –), pair of half-immersed ellipses with draft $0.1h$ (– · – · –)

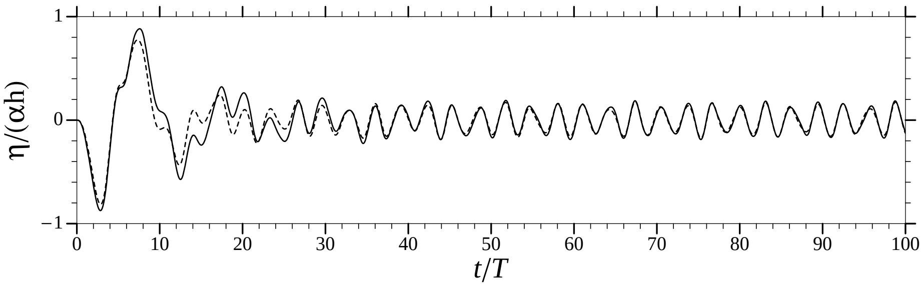

Figure 5: Free-surface elevation $\eta(0, t)$ generated by the forced motion of the trapping structure (———) and the half-immersed circles (– – – –).

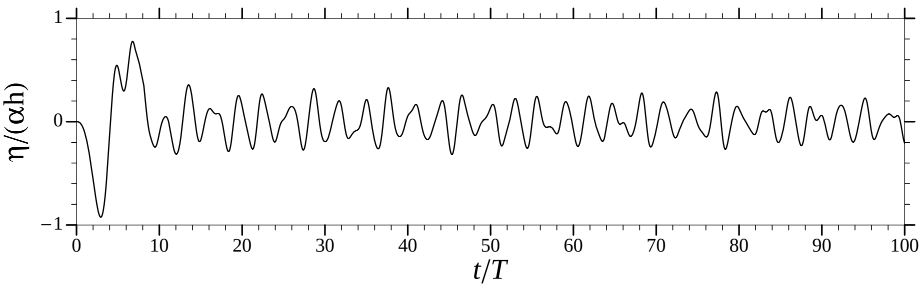

Figure 6: Free-surface elevation $\eta(0, t)$ generated by the forced motion of the half-immersed ellipses.

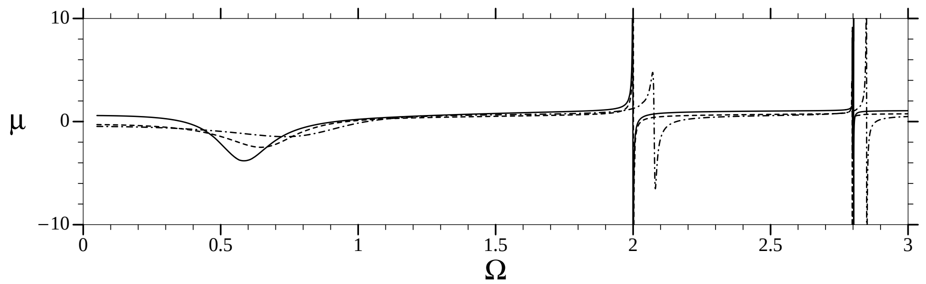

Figure 7: Non-dimensional heave added mass μ vs. non-dimensional frequency $\Omega = \omega/\sqrt{g/h}$ for the structures shown in figure 4: trapping structure (———), pair of half-immersed circles (– – – –), pair of half-immersed ellipses (– · – · –)

Question by : J.N. Newman
My guess is that the last line in Table 1 is simply a consequence of the standing wave "sloshing" resonance, and that there is no "pure" trapping mode for this geometry. Do you agree?

Author's reply:
"Sloshing" and "pure trapping" are very closely related. On the basis of numerical calculations it is very difficult (impossible?) to distinguish between the two. It is perhaps unlikely that the last line of Table 1 corresponds to a trapped mode, but I can not be ruled out.

Question by : H. Bingham
These trapping structures do not radiate at the trapped-mode frequency, yet as we saw in the previous talk they have non-zero damping. Can you explain this?

Author's reply:
A fixed trapping supports a free oscillation that does not radiate. If the structure is forced to oscillate at the trapped mode frequency (in a suitable mode) then the frequency-domain solution does not exist. The previous talk was concerned with the limit as the trapped-mode frequency is approached
In the time-domain, a trapping structure forced to oscillate at the trapped frequency does radiate waves at that frequency (see the paper by McIver, McIver & Zhang presented at the 17th IWWWFB). The non-zero limit of the damping refered to above may be related to this, but this has not been confirmed.

Question by : D.V. Evans
The behaviour of the free surface in time is very similar for the trapping structure and for a pair of semicircles. How long is it before the oscillations in the case of the semicircle tends to zero?

Comment by : D.V. Evans
You suggested that trapped modes did not exist for "real" bodies. But Shipway (Ph.D thesis University of Bristol 2003) has obtained a wide range of trapped modes for the case of two vertical partially immersed concentric shells in finite water depth

Author's reply:
The decay rate can be estimated from the numerical calculations. For the two semicircles the amplitude of the oscillation at frequency $\Omega \approx 2$ decreases by about 2.5% in a time interval 100T.

Question by : M. Meylan
How did you solve the time domain problem? Do you think that solving for the complex roots would eliminate this theory?

Author's reply:
The time-domain problem was solved using a time-stepping technique that is described briefly in the paper by McIver, McIver & Zhang presented at the 17^{th} IWWWFB.
Calculations of the "complex roots" is certainly useful. For example, the position of a root yields the decay rate of a near resonant oscillation in the time domain. An estimate of the root position can be obtained from the added mass, but a more refined calculation procedure would give more complete information.

Question by : R. Porter
You demonstrated the close connection between zeros of transmission for a single body and trapping between pairs of structures. Have you computed the transmission coefficient for the non-trapping structure (circle+ellipse) and confirmed the non-existence (or otherwise) of zeros of transmission?

Author's reply:
For the circle and ellipse the transmission coefficient is a monotically decreasing function of frequency that has no zero. This highlights the special nature of the trapping structure.