Infinite depth interaction theory for arbitrary bodies with application to wave forcing of many ice floes

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1 Introduction

The scattering of water waves by floating or submerged bodies is a complex research topic. While analytic solutions have been found for simplified problems (especially for simple geometries or in two dimensions) the full three-dimensional linear diffraction problem can only be solved by numerical methods involving the discretisation of the body's surface. The resulting linear system of equations has a dimension equal to the number of unknowns used in the discretisation of the body.

If more than one body is present, all bodies will scatter the incoming waves. Therefore, the scattered wave from one body will be incident upon all the others and, given that they are not too far apart, this will notably change the total incident wave upon them. Therefore, the diffraction calculation must be conducted for all bodies simultaneously. Since each body must be discretised this can lead to a very large number of unknowns. However, the scattered wavefield can be represented in an eigenfunction basis with a comparatively small number of unknowns.

The first interaction theory that was not based on an approximation was developed by Kagemoto & Yue (1986). Kagemoto and Yue found an exact algebraic method to solve the linear wave problem for vertically non-overlapping bodies in water of finite depth. The interaction of the bodies was accounted for by taking the scattered wave of each body to be the incident wave upon all other bodies (in addition to the ambient incident wave). Doing this for all bodies Kagemoto & Yue were able to solve for the coefficients of the scattered wavefields of all bodies simultaneously. The only difficulty with this method was that the solutions of the single diffraction problems had to be available in the cylindrical eigenfunction expansion of an outgoing wave. Kagemoto & Yue therefore only solved for axisymmetric bodies because the single diffraction solution for cylinders was available in the required representation.

The extension of the Kagemoto and Yue scattering theory to bodies of arbitrary geometry was performed by Goo & Yoshida (1990) who found a general way to solve the single diffraction problem in the required cylindrical eigenfunction representation. They used the representation of the finite depth free surface Green's function in the eigenfunction expansion of cylindrical outgoing waves centred at an arbitrary point of the water surface (above the body's mean centre position in this case) given by Fenton (1978).

The development of the Kagemoto and Yue interaction theory was motivated by problems in off-shore engineering. However, the theory can also be applied to the geophysical problem of wave scattering by ice floes. A method of solving for the wave response of a single ice floe of arbitrary geometry in water of infinite depth was suggested by Meylan (2002). The ice floe was modelled as a floating, flexible thin plate of shallow draft. However, to understand wave propagation and scattering in the by many ice floes, in the Marginal Ice Zone (MIZ) for example, we need to understand the way in which large numbers of interacting ice floes scatter waves. For this reason, we require an interaction theory. While we could use the Kagemoto and Yue interaction theory, their theory requires that the water depth is finite. Since the water is deep in the MIZ this means that we must set the finite depth large in order to be able to apply their theory. For this reason, we develop the equivalent interaction theory to Kagemoto and Yue except for infinite depth. Furthermore, because of the complicated geometry of an ice floe this interaction theory must be for bodies of arbitrary geometry.

In the first part Kagemoto and Yue's interaction theory is extended to water of infinite depth. We represent the incident and scattered potentials in the cylindrical eigenfunction expansions and we use an analogous infinite depth Green's function to the one used by Goo & Yoshida, which is given by Peter & Meylan (2002). We show how the infinite depth diffraction transfer matrices can be obtained with the use of this Green's function and how the rotation of a body about its mean centre position in the plane can be accounted for without recalculating the diffraction transfer matrix.

In the second part, using Meylan's result, the motion and scattering of many interacting ice floes is calculated and presented. For two square interacting ice floes the convergence of the method obtained from the developed interaction theory is compared to the result of the full diffraction calculation. Solutions of more than two interacting ice floes are presented as well. We also compare the convergence of the finite depth and infinite depth methods in deep water. It is well known that diffraction calculations in water of infinite depth require less numerical effort than the equivalent finite depth calculations with the depth chosen sufficiently deep. A similar advantage is true for the interaction theory as well.

2 The interaction method

Kagemoto & Yue (1986) developed an interaction theory for vertically non-overlapping axisymmetric structures in water of finite depth. In this section we will extend their theory to bodies of arbitrary geometry in water of infinite depth.

The equations of motion for the water are derived from the linearised inviscid theory. Only fixed radian frequencies ω are considered so the time-dependence of the water velocity potential is factored out, $\Phi(\mathbf{y},t) = \mathbb{R}e\,\{\phi(\mathbf{y})\mathrm{e}^{-\mathrm{i}\omega t}\}$. The water surface is assumed at $z=0$.

The problem consists of N vertically non-overlapping bodies, denoted by Δ_j, which are sufficiently far apart that there is no intersection of the smallest cylinder which contains each body with any other body. Let (r_j,θ_j,z) be the local cylindrical coordinates of the jth body and $\alpha = \omega^2/g$ where g is the acceleration due to gravity. The scattered potential of body Δ_j can then be expanded in cylindrical eigenfunctions,

$$\phi_j^{\mathrm{S}}(r_j,\theta_j,z) = \mathrm{e}^{\alpha z} \sum_{\nu=-\infty}^{\infty} A_{0\nu}^j H_\nu^{(1)}(\alpha r_j)\mathrm{e}^{\mathrm{i}\nu\theta_j}$$

$$+ \int_0^\infty \big(\cos\eta z + \frac{\alpha}{\eta}\sin\eta z\big) \sum_{\nu=-\infty}^{\infty} A_\nu^j(\eta) K_\nu(\eta r_j)\mathrm{e}^{\mathrm{i}\nu\theta_j}\,\mathrm{d}\eta,$$

with discrete coefficients $A_{0\nu}^j$ for the propagating modes and coefficient functions $A_\nu^j(\cdot)$ for the decaying modes. The radial eigenfunctions $H_\nu^{(1)}$ and K_ν are the Hankel function of the first kind and the modified Bessel function of the second kind respectively, both of order ν. Analogously, the incident potential upon body Δ_j can be expanded in cylindrical eigenfunctions. In this case, the propagating and decaying radial eigenfunctions are given by J_μ and I_μ respectively, the Bessel function and the modified Bessel function respectively, both of the first kind and order μ. The coefficients of the incident potential will be denoted with $D_{0\mu}^j$ and $D_\mu^j(\cdot)$. To simplify notation, from now on $\psi(z,\eta)$ will denote the vertical eigenfunctions corresponding to the decaying modes, $\psi(z,\eta) = \cos\eta z + \alpha/\eta\,\sin\eta z$.

2.1 The interaction theory

Following the ideas of Kagemoto & Yue (1986), a system of equations for the unknown coefficients of the scattered wavefields will be developed. This system of equations is based on transforming the scattered potential of Δ_j into an incident potential upon Δ_l ($j \neq l$). Doing this for all bodies simultaneously and relating the incident and scattered potential for each body, a system of equations for the unknown coefficients will be derived.

The scattered potential ϕ_j^{S} of body Δ_j needs to be represented in terms of the incident potential ϕ_l^{I} upon Δ_l, $j \neq l$. This can be accomplished by using Graf's addition theorem for Bessel functions given in Abramowitz & Stegun (1964) which is valid provided that $r_l < R_{jl}$. Here and in the sequel, (R_{jl},ϑ_{jl}) are the polar coordinates of the mean centre position of Δ_l in the coordinate system of Δ_j. The ambient incident wavefield ϕ^{In} can also be expanded in the eigenfunctions corresponding to the incident wavefield upon Δ_l. Let $D_{l0\mu}^{\mathrm{In}}$ denote the coefficients of this ambient incident wavefield corresponding to the propagating modes and $D_{l\mu}^{\mathrm{In}}(\cdot)$ to the decaying modes (which are identically zero) of the incoming eigenfunction expansion for Δ_l. The coefficients of the total incident potential upon Δ_l can now be expressed as

$$D_{0\mu}^l = D_{l0\mu}^{\mathrm{In}} + \sum_{\substack{j=1\\ j\neq l}}^{N} \sum_{\nu=-\infty}^{\infty} A_{0\nu}^j H_{\nu-\mu}^{(1)}(\alpha R_{jl})\mathrm{e}^{\mathrm{i}(\nu-\mu)\vartheta_{jl}},$$

$$D_\mu^l(\eta) = D_{l\mu}^{\mathrm{In}}(\eta) + \sum_{\substack{j=1\\ j\neq l}}^{N} \sum_{\nu=-\infty}^{\infty} A_\nu^j(\eta)(-1)^\mu K_{\nu-\mu}(\eta R_{jl})\mathrm{e}^{\mathrm{i}(\nu-\mu)\vartheta_{jl}}.$$

In general, it is possible to relate the total incident and scattered partial waves for any body through the diffraction characteristics of that body in isolation. For the propagating and the decaying modes respectively, the scattered potential can be related to the incident potential by diffraction transfer operators B acting as follows,

$$A_{0\nu}^l = \sum_{\mu=-\infty}^{\infty} B_{l\nu\mu}^{\mathrm{pp}} D_{0\mu}^l + \int_0^\infty \sum_{\mu=-\infty}^{\infty} B_{l\nu\mu}^{\mathrm{pd}}(\xi) D_\mu^l(\xi)\,\mathrm{d}\xi,$$

$$A_\nu^l(\eta) = \sum_{\mu=-\infty}^{\infty} B_{l\nu\mu}^{\mathrm{dp}}(\eta) D_{0\mu}^l + \int_0^\infty \sum_{\mu=-\infty}^{\infty} B_{l\nu\mu}^{\mathrm{dd}}(\eta;\xi) D_\nu^l(\xi)\,\mathrm{d}\xi.$$

The superscripts p and d are used to distinguish between propagating and decaying modes, the first superscript denotes the kind of scattered mode, the second one the kind of incident mode.

Implying a suitable truncation, the four different diffraction transfer operators can be represented by matrices which can be assembled in a big matrix $\mathbf{B}_l$, the infinite depth diffraction transfer matrix. Truncating the coefficients accordingly, $\mathbf{a}^l$ as the vector of the coefficients of the scattered potential of body Δ_l as well as $\mathbf{d}_l^{\mathrm{In}}$ as the vector of coefficients of the ambient wavefield and making use of a matrix $\mathbf{T}_{jl}$ accounting for the coordinate transformation, a linear system of equations follows,

$$\mathbf{a}_l = \hat{\mathbf{B}}_l\Big(\mathbf{d}_l^{\mathrm{In}} + \sum_{\substack{j=1\\ j\neq l}}^{N} {}^{\mathrm{t}}\mathbf{T}_{jl}\,\mathbf{a}_j\Big), \quad l = 1\ldots N. \tag{1}$$

The matrix $\hat{\mathbf{B}}_l$ denotes the infinite depth diffraction transfer matrix $\mathbf{B}_l$ in which the elements associated with decaying scattered modes have been multiplied with the appropriate integration weights depending on the discretisation of the continuous variable.

2.2 The diffraction transfer matrix for bodies of arbitrary geometry

The diffraction transfer matrices $\mathbf{B}_j$ relate the incident and the scattered potential for a body Δ_j in isolation. Their elements, $(\mathbf{B}_j)_{pq}$, are given by the coefficients of the pth partial wave of the scattered potential due to a single unit-amplitude incident wave of mode q upon Δ_j.

The calculation of the diffraction transfer matrices for bodies of arbitrary geometry has been performed by Goo & Yoshida (1990) in the case of finite depth. They used the standard method of transforming the single diffraction boundary value problem to an integral equation over the immersed surface of the body using a Green's function. To obtain cylindrical eigenfunction expansions of the potential, Goo & Yoshida utilised the representation of the free surface finite depth Green's function given by Fenton (1978).

To calculate the diffraction transfer matrix in infinite depth, we require the representation of the infinite depth free surface Green's function in cylindrical eigenfunctions, given by Peter & Meylan (2002),

$$G(\mathbf{r};\zeta) = \frac{\mathrm{i}\alpha}{2}\,\mathrm{e}^{\alpha(z+c)} \sum_{\nu=-\infty}^{\infty} H_\nu^{(1)}(\alpha r) J_\nu(\alpha s) \mathrm{e}^{\mathrm{i}\nu(\theta-\varphi)} + \int_0^\infty \psi(z,\eta) \frac{\eta^2\,\pi^{-2}}{\eta^2+\alpha^2} \psi(c,\eta) \sum_{\nu=-\infty}^{\infty} K_\nu(\eta r) I_\nu(\eta s) \mathrm{e}^{\mathrm{i}\nu(\theta-\varphi)}\,\mathrm{d}\eta,$$

$r > s$, where $\mathbf{r} = (r,\theta,z)$ and $\zeta = (s,\varphi,c)$.

We assume that we have represented the scattered potential in terms of the source strength distribution ς so that the scattered potential can be written as

$$\phi_j^{\mathrm{S}}(\mathbf{r}) = \int_{\Gamma_j} G(\mathbf{r},\zeta)\,\varsigma_j(\zeta)\,\mathrm{d}\sigma_\zeta, \quad \mathbf{r} \in D, \tag{2}$$

where D is the volume occupied by the water and Γ is the immersed surface of the body. Substituting the eigenfunction expansion of the Green's function into (2), the scattered potential can be written in the cylindrical eigenfunction representation as long as $r > s$. This restriction implies that the eigenfunction expansion is only valid outside the escribed cylinder of the body. The elements of the diffraction transfer matrix are therefore given by

$$(\mathbf{B}_j)_{pq} = \frac{\mathrm{i}\alpha}{2} \int_{\Gamma_j} \mathrm{e}^{\alpha c} J_p(\alpha s) \mathrm{e}^{-\mathrm{i}p\varphi} \varsigma_q^j(\zeta)\,\mathrm{d}\sigma_\zeta,$$

$$(\mathbf{B}_j)_{pq} = \frac{1}{\pi^2} \frac{\eta^2}{\eta^2+\alpha^2} \int_{\Gamma_j} \psi(c,\eta) I_p(\eta s) \mathrm{e}^{-\mathrm{i}p\varphi} \varsigma_q^j(\zeta)\,\mathrm{d}\sigma_\zeta,$$

for the propagating and the decaying modes respectively, where $\varsigma_q^j(\zeta)$ is the source strength distribution due to unit-amplitude incident potentials of mode q.

For a non-axisymmetric body, a rotation about the mean centre position in the (x,y)-plane will result in a different diffraction transfer matrix. However, the additional angular dependence caused by the rotation of the body can be factored out of the elements of the diffraction transfer matrix. The elements of the diffraction transfer matrix corresponding to the body rotated by the angle β, $\mathbf{B}_j^\beta$, are given by $(\mathbf{B}_j^\beta)_{pq} = (\mathbf{B}_j)_{pq}\,\mathrm{e}^{\mathrm{i}(q-p)\beta}$.

3 Results

In this section we will present some calculations using the interaction method in finite depth and infinite depth and the full diffraction method in finite and infinite depth. These will be based on calculations for ice floes and will in no way be exhaustive. The wave scattering of a single ice floe of arbitrary geometry was described by Meylan (2002). The full diffraction calculation for many ice floes can be derived straightforwardly from his work. Based on Meylan's results we begin with a convergence comparison involving two square interacting ice floes on deep water which aims to illustrate and compare the various methods. A result for more than two ice floes is presented after. At first, however, some numerical considerations have to be made.

For the numerical calculations truncation parameters have to be introduced. Truncating the infinite sums in the eigenfunction expansion of the outgoing water velocity potential for infinite depth and discretising the integration by defining a set of nodes η_m with weights h_m the potential can be approximated by

$$\phi(r,\theta,z) = \mathrm{e}^{\alpha z} \sum_{\nu=-T_H}^{T_H} A_{0\nu} H_\nu^{(1)}(\alpha r) \mathrm{e}^{\mathrm{i}\nu\theta} + \sum_{m=1}^{T_R} h_m\,\psi(z,\eta_m) \sum_{\nu=-T_K}^{T_K} A_\nu(\eta_m) K_\nu(\eta_m r) \mathrm{e}^{\mathrm{i}\nu\theta}. \tag{3}$$

In the following, the integration weights are chosen to represent the mid-point quadrature rule. In finite depth, an analogous truncation can be performed. T_R then denotes the number of roots of the dispersion relation used in the calculations. In water of finite depth, the depth can also be considered a free parameter as long as it is chosen large enough to account for deep water.

3.1 Convergence comparison

We will present a convergence comparison that aims to illustrate the performance of the interaction theory and to compare the performance of the finite and infinite depth interaction methods in deep water. Results of full diffraction calculations serve as references.

To compare the results, a measure of the error from the full diffraction calculation is used, $(E_2)^2 = \int_\Delta \left|w_i(\mathbf{x}) - w_f(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}$, where w_i and w_f are the solutions of the interaction method and the corresponding full diffraction calculation respectively.

The most challenging situation for the interaction theory is when the bodies are close together. An interesting and representative arrangement is where the second ice floe is located closely behind the first. This arrangement is used in this illustration. Tests with other arrangements give similar results.

Since the choice of the number of propagating modes and angular decaying components affects the finite and infinite depth methods in similar ways, the dependence on these parameters will not be further presented. Thorough convergence test have shown that in the settings investigated here, it is sufficient to choose T_H to be 11 and T_K to be 5. We will now compare the convergence of the infinite depth and the finite depth methods if T_H and T_K are fixed (with the previously mentioned values) and T_R is varied.

The exact positions of the ice floes in this test are given by $O_1 = (-1.4, 0)$ and $O_2 = (1.4, 0)$. Both ice floes have non-dimensionalised stiffness $\beta = 0.02$ and mass $\gamma = 0.02$ (using Meylan's non-dimensionalisation). The wavelength of the ambient incident wave is 2, the side length of each square ice floe is one wavelength. It should be noted that for these ice floe parameters, the water can be considered deep if the depth is greater than 1.5 (Fox & Squire, 1994). The ambient wavefield is of unit amplitude and propagates in the x-direction. In the full diffraction calculation the ice floes are discretised in $24 \times 24 = 576$ square elements.

Figure 1 shows the solutions in the case of water of infinite depth. To illustrate the effect on the water in the vicinity as well, its displacement is also shown. A close view and a far view from above (plan view) are shown. In the plan view, the discretisation mesh has been removed and the greyscale has been interpolated for visibility.

Figure 2 shows the convergence for the infinite depth method and the finite depth method with depth 3. As can be seen, the convergence is quite similar. However, it must be noted that the convergence of the infinite depth method depends on the choice of nodes (an average set is chosen in this comparison) while the convergence of the finite depth method varies with the chosen depth.

3.2 Multiple ice floe results

The interaction method can now be used to calculate the displacement of many interacting ice floes on water of infinite depth. Figure 3 shows the displacements of five square interacting ice floes. The parameters are chosen identically to those in the convergence comparison.

4 Summary

The finite depth interaction theory developed by Kagemoto & Yue (1986) has been extended to water of infinite depth. Furthermore, using the eigenfunction expansion of the infinite depth free surface Green's function we were able to calculate the diffraction transfer matrices for bodies of arbitrary geometry. We also showed how the diffraction transfer matrices can be calculated efficiently for different orientations of the body.

The convergence of the infinite depth interaction method is similar to that of the finite depth method. However, for the infinite depth method the discretisation of the continuous variable can be chosen optimally as opposed to the finite depth method where the summation weights are given by the roots of the dispersion relation. The infinite depth interaction method has two further advantages. Firstly, it requires the infinite depth single diffraction solution which is easier to compute than the finite depth solution. The second advantage is that there is no danger that the depth may have been chosen too shallow to approximate infinitely deep water.

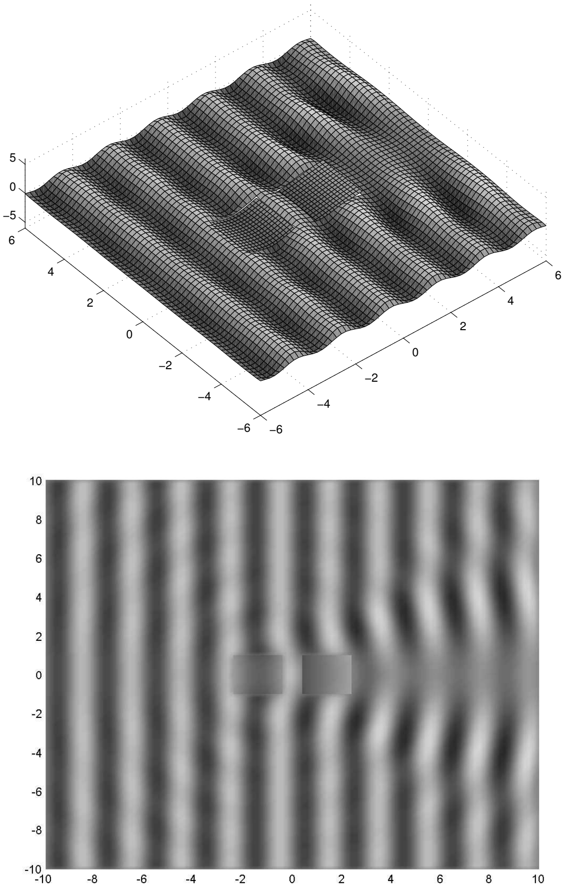

Fig. 1: Interacting ice floes and their water in the vicinity, close view and plan view

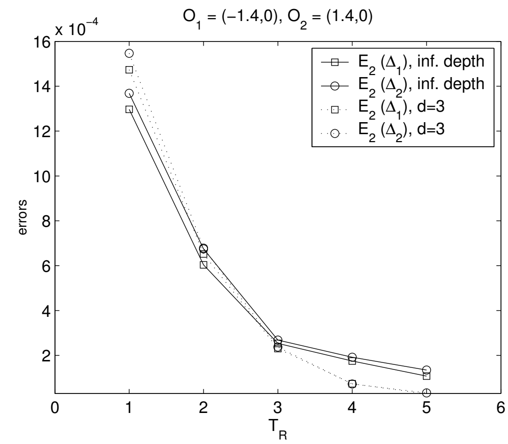

Fig. 2: Development of the errors as T_R is increased

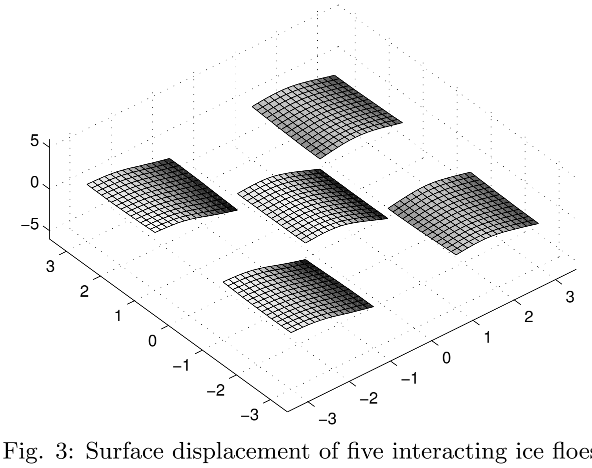

Fig. 3: Surface displacement of five interacting ice floes

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Question by : M. Kashiwagi

1. In the infinite depth case, the infinite integral must be evaluated in place of the summation with respect to the wavenumber. How did you evaluate this integral?
2. How are you going to treat the case of a great number of ice floes (the order of several thousands)?

Author's reply:

1. Since the integral decays quickly it is sufficient to integrate up to a small positive number. This is done with standard numerical quadrature methods.
2. Our aim is to be able to solve for a sufficient number of ice floes to observe convergence. We have not yet run these computations, but we hope to obtain convergence with hundreds of ice floes.
