Optical theorem in atomic physics and wave power

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In atomic physics, when a plane wave of amplitude 1 interacts with an atom or nucleus, the amplitude at distance R and angle θ is taken as

$$A(\theta) = (1/kR)f(\theta) \tag{1}$$

where $k = 2\pi/\lambda$ and $f(\theta)$ is called the scattering amplitude. For real f the radiation is in phase with the incident wave. The "optical theorem" states that the total cross-section (i.e. the effective combined target area for scattering and capture) is

$$\sigma_{tot} = 4\pi\lambdabar^2 \, \Im\{f(0)\} \tag{2}$$

with $\lambdabar = \lambda/2\pi$.

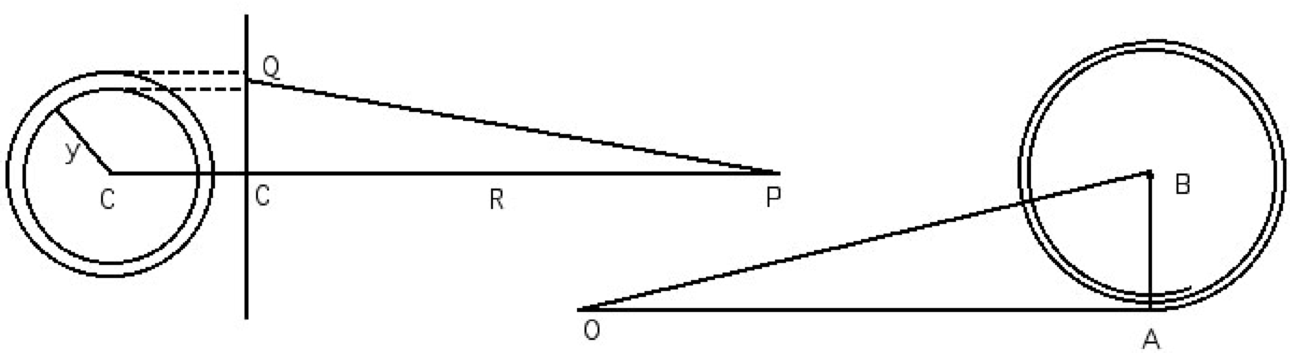

Figure 1: Fraunhofer spiral

To understand this formula, consider a plane of atoms transverse to the incoming wave, as shown in Fig. 1, with n atoms/cm^2, and sum over the wavelets reaching point P on the axis from annular elements of radius y to $y + dy$, using the Fraunhofer spiral. The phase delay for this annulus, relative to the wave from the centre, is $\theta = \pi y^2/R\lambda$, while the elementary vector at P corresponding to this annulus has length $dV = 2\pi y dy \times nf(0)/kR$. As $d\theta \propto dV$, the vectors add to a circle of radius $AB = dV/d\theta = 2\pi n\lambdabar^2 f(0)$.

If P is far from the plane of atoms, only the forward scattering amplitude $f(0)$ is relevant. The elements far from the axis contribute gradually less and less (because of their increasing distance from P) so adding over the whole plane gives a slowly diminishing spiral and the final vector is just the

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vector AB of length $2\pi n\bar{\lambda}^2 f(0)$. The vital point is that although all the atoms radiate in phase with the incoming wave, the vector AB is delayed 90^o relative to the original wave at P, because on average the path length to P is slightly increased. Adding AB to the original vector OA of length 1, the resultant OB is slightly delayed in phase. This is the origin of refractive index.

If there are ρ, atoms per cm^3 and the layer of atoms is t thick, the phase shift is

$$(\mu - 1)t/\lambda = 2\pi\bar{\lambda}^2 f(0)\rho t$$

so

$$\mu - 1 = 4\pi^2\rho\bar{\lambda}^3 f(0)$$

is determined by the real part of the forward scattering amplitude. This tells us how to make refracting lenses in the sea, using a field of undamped resonant scatterers!

If $f(0)$ is imaginary we get an extra phase shift of 90^o and the vector AB subtracts from the incoming vector OA: the wave is attenuated. For small n,

$$OB^2 = \{1 - 2\pi n\bar{\lambda}^2\ \Im\{f(0)\}\}^2 = 1 - n\sigma_{tot}$$

where σ_{tot} is the total cross-section per atom for interaction with the incoming wave: Eqn.(1) follows.

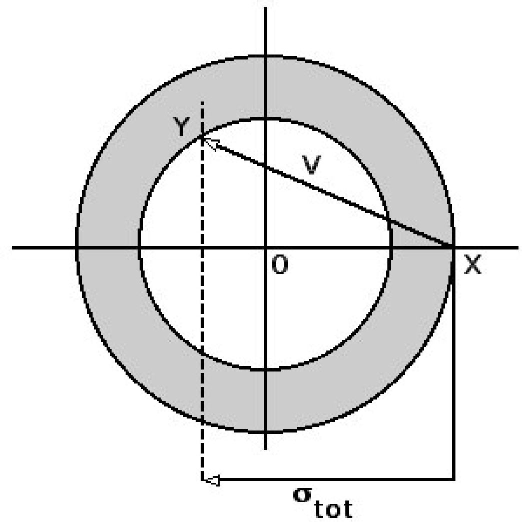

Figure 2: η circle, radius 1

σ_{tot} is the effective area for one atom interacting with the incoming wave; it includes scattering and capture.

$$\sigma_{tot} = \sigma_{scat} + \sigma_{cap}.$$

If the size of the interacting body is much less than λ, the scattered amplitude is isotropic and we can write $f = \frac{1}{2}i(1-\eta)$. Then

$$\sigma_{tot} = 2\pi\bar{\lambda}^2\ \Re\{1-\eta\} \tag{3}$$

But by integrating (1) over the whole solid angle 4π,

$$\sigma_{scat} = 4\pi\bar{\lambda}^2 f^2 = \pi\bar{\lambda}^2|1-\eta|^2 \tag{4}$$

so

$$\sigma_{cap} = \sigma_{tot} - \sigma_{scat} = \pi\bar{\lambda}^2(1 - |\eta|^2) \tag{5}$$

From this we conclude that $|\eta| \leq 1$. Fig. 2 shows the circle in the complex plane which includes all possible values of η. If η is represented by the point Y, the shaded area is proportional to σ_{cap}. The square of the vector XY determines σ_{scat}, while twice its horizontal projection gives σ_{tot}. Maximum power is captured when $\eta = 0$ and in this case $\sigma_{cap} = \sigma_{scat} = \frac{1}{2}\sigma_{tot}$.

For water waves in two dimensions, similar arguments apply. Consider an incoming plane wave of unit amplitude incident on a line of objects, n per unit length, each radiating an amplitude

$$A(\theta) = (\lambda/R)^{1/2} f(\theta) \tag{6}$$

As before the phase delay for a signal from Q (fig. 1) is $\pi y^2/R\lambda = \pi v^2/2$ where $v = y\sqrt{2/R\lambda}$ while the length of the elementary vector at P corresponding to the line element dy at Q is

$$n(\lambda/R)^{1/2} f(\theta)\sqrt{R\lambda/2}\ dv = n\lambda f(0) dv/\sqrt{2}.$$

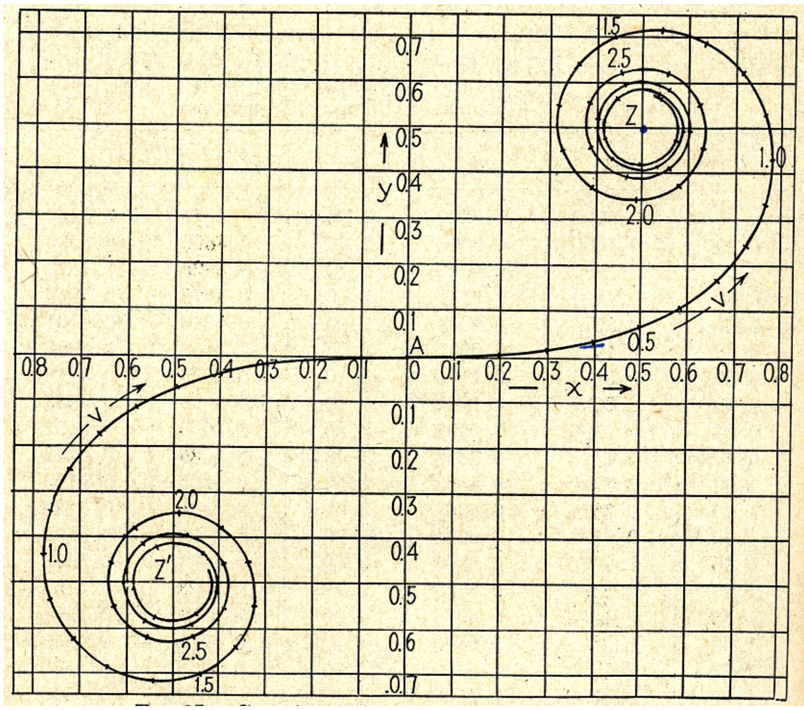

Figure 3: Cornu spiral

This combination of vector length and phase generates the Cornu spiral (fig. 3) and the resultant vector has components C (real) and S (imaginary) given by the Fresnel integrals [1]

$$C = n\lambda f(0)/\sqrt{2}\int_{-\infty}^{\infty}\cos(\pi v^2/2)dv \tag{7}$$

$$S = n\lambda f(0)/\sqrt{2}\int_{-\infty}^{\infty}\sin(\pi v^2/2)dv \tag{8}$$

Both integrals are equal to 1, so the resultant vector has length $n\lambda f(0)$ but its phase delay is only 45^o. Now recall that the individual bodies, moving in phase with the incident wave, will radiate a Bessel function. At large distances the asymptotic form is $\cos(kR - \pi/4)$ which gives an extra phase delay of 45^o so the resultant vector is, as before, delayed 90^o on the incident wave at P; the real part of $f(0)$ will lead to refraction, while the imaginary part gives the total interaction width W_{tot} which includes scattered power W_{scat} and captured power W_{cap}.

$$W_{tot} = 2\lambda\ \Im\{f(0)\} \tag{9}$$

If the radiated wave is isotropic, $f = f(\theta) = 1$, and we can use the ansatz

$$f = \frac{i}{2\pi}(1-\eta) \tag{10}$$

which gives

$$W_{tot} = 2\bar{\lambda}\ \Re\{1-\eta\} \tag{11}$$

$$W_{scat} = 2\pi\lambda|f|^2 = \bar{\lambda}|1-\eta|^2 \tag{12}$$

$$W_{cap} = \bar{\lambda}(1-|\eta|^2) \tag{13}$$

and fig. 2 will again apply.

More generally, when the polar diagram is not isotropic,

$$W_{scat} = \lambda \int_0^{2\pi} |f(\theta)|^2 \, d\theta \tag{14}$$

and subtracting this from eqn(9) gives

$$W_{cap} = 2\lambda \, \Im\{f(0)\} - \lambda \int_0^{2\pi} |f(\theta)|^2 \, d\theta \tag{15}$$

Note that $f(\theta)$ enters linearly in the first term and squared in the second, so by varying the magnitude of $f(\theta)$ keeping the same functional form, W_{cap} can be maximized and the optimum value is

$$W_{cap}^{max} = \lambda \frac{[\Im\{f(0)\}]^2}{\int_0^{2\pi} |f(\theta)|^2 \, d\theta} \tag{16}$$

This formula has been found independently for particular cases by Newman [2] and Evans [3]. The more general result was derived by Farley [4]. Eqn(16) is completely independent of the mechanism of wave capture and applies to any combination of vibrational modes in the device; the capture width depends only on the polar diagram and the phase of the radiated waves.

For example, for a small heaving buoy the radiation is isotropic, $f_0(\theta) = 1$, and the best capture width is $\lambda/2\pi$ [5]. For a small object moving in surge or pitch however $f_1(\theta) = \cos(\theta)$ and the capture width rises to $2 \times \lambda/2\pi$. One may combine these modes to give

$$f(\theta) = f_0(\theta) + \beta f_1(\theta)$$

and vary β to get the best result; one finds $\beta = 2$ which gives $W_{cap} = 3 \times \lambda/2\pi$ in agreement with Falnes [6].

It is clear from eqn(16) that an ideal attenuator should radiate only in the forward direction. The more concentrated the polar diagram, the greater the capture width. But antenna theory tells us that to have a narrow polar diagram, the radiator must be several wavelengths long, either in the beam direction or transverse to it. Furthermore, to concentrate the beam in the forward direction a traveling wave should be excited on the attenuator.

References

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