FEM-Wagner coupling for hydroelastic impact problems

Guéret R.[1], **Korobkin A.A.**[2] **& Malenica Š.**[1]

(1) Bureau Veritas - DR,Paris, France (renaud.gueret@bureauveritas.com, sime.malenica@bureauveritas.com)
(2) Lavrentyev Institute of Hydrodynamics, Novosibirsk,Russia (kaaa@hydro.nsc.ru)

Introduction

Unsteady two-dimensional problem of elastic structure impact onto a liquid free surface is considered within the Wagner approximation. Elastic deflection of the structure during water impact is described by Euler beam equation with additional support conditions.
This problem has been intensively studied decomposing the deflection in normal modes of the structure vibration in air. This method is well suited when considering an homogeneous beam or simple geometric shapes but looses all the advantages of the analytical developments when the body is of a more general shape. There is a possibility to use the advantages of the Wagner approach in combination with the finite element method which is a main tool in the structural analysis. This combined method can deal with complicated elastic structures treating the hydrodynamic part of the coupled hydroelastic problem in a way similar to that for the homogeneous one. The development of the finite element method for the structural part in combination with the Wagner approach is the subject of the present study.
Advantages of the present technique are illustrated by analysis of the symmetric problem of elastic wedge entry. The obtained numerical results are compared with those by the normal mode method in the case of wedge platings of constant thickness. Very good agreement between these two methods has been found.

Mathematical model

The problem of elastic wedge impact is formulated as follows. Initially an elastic wedge touches horizontal free surface of the liquid at a single point ($x' = 0$, $y' = 0$) and starts to move down at instant $t' = 0$ with a constant velocity V. The side walls of the wedge are modeled as simply supported identical Euler beams. The beams are of variable thickness. In this case the flow caused by the wedge impact is symmetric with respect to the line $x' = 0$. In symmetric case, normal deflection of the beam is denoted by $w'(s', t')$, where s' is the coordinate along the initially flat side walls, $s' = 0$ corresponds to the wedge tip and $s' = L$ to the beam end point. For small deadrise angle of the wedge, we have $s' \approx x'$. Position of the wedge side walls is described by the equation $y' = |x'| \tan\gamma + w'(x', t') - Vt$, $|x'| < L\cos\gamma$, where γ is the deadrise angle of the equivalent rigid wedge and L is the length of the side walls.

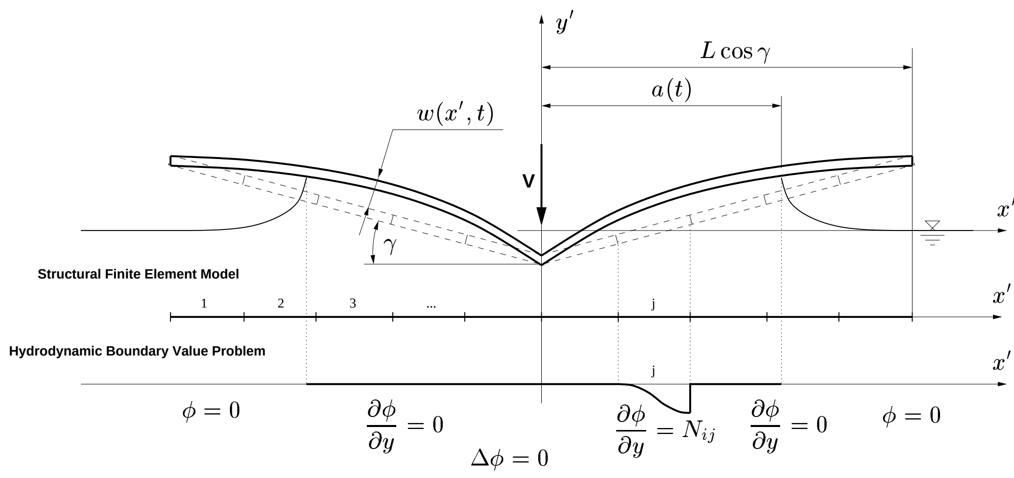

Figure 1: *Basic configuration and definitions.*

The beam length L is taken as the length scale and the impact velocity V as the velocity scale of liquid flow. The quantity $T = (L/V)\sin\gamma$ is taken as the time scale and the product $L\sin\gamma$ as the displacement scale. The product VL is taken as the scale of the velocity potential and the quantity $\rho V^2/\sin\gamma$ as

the hydrodynamic pressure scale. Wagner theory is applied to the entry problem of a wedge with small deadrise angle γ [1].
Within the adopted Wagner approach two-dimensional and potential flow caused by beam impact is described in non-dimensional variables by the velocity potential $\varphi(x,y,t)$ which satisfies the following equations:

$$\varphi_{xx}+\varphi_{yy}=0, \qquad p=-\varphi_t \qquad (y<0), \tag{1}$$

$$p=0, \qquad \eta_t=\varphi_y \qquad (y=0, \quad x>a(t) \quad \text{and} \quad x<-b(t)), \tag{2}$$

$$\varphi_y=-1+w_t \qquad (y=0, \quad -b(t)<x<a(t)), \tag{3}$$

$$\varphi\to 0 \qquad (x^2+y^2\to\infty). \tag{4}$$

The point with coordinates $(-b(t),0)$ moves along the boundary and corresponds to the left edge of the contact region between the entering body and the liquid, and the point with coordinates $(a(t),0)$, which moves to the right, corresponds to the right edge of the contact region. The functions $a(t)$ and $b(t)$ are unknown in advance and have to be determined together with the solution. In the symmetric case $b(t)=a(t)$. Finally, $p(x,y,t)$ is the hydrodynamic pressure, which is defined by the linearized Bernoulli's equation.
The beam deflection $w(x,t)$ is governed by the equations

$$m(x)\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)=p(x,0,t) \qquad (-1<x<1, \quad t>0), \tag{5}$$

$$w=w_{xx}=0 \qquad (x=-1,\ x=0,\ x=1, \quad t\geq 0), \tag{6}$$

$$w=w_t=0 \qquad (-1<x<1, \quad t=0) \tag{7}$$

where $x=x'/L$, $m(x)=m'(xL)/(\rho L)$ is the non-dimensional mass distribution, E is the Young's modulus of elasticity, $EI(x)=EI'(xL)\sin^2\gamma/(\rho V^2L^3)$ is the non-dimensional rigidity of the beam and $I'(x')$ is the moment of inertia of the cross section. Edge conditions (6) are chosen to represent the simply supported beam and can be easily replaced for other cases.
The additional condition, which is known as the Wagner condition, states that at the contact points $x=a(t)$ and $x=-b(t)$ the elevations of the free surface are equal to the vertical coordinates of the deformed plate at these points. This condition leads to the following equations for $a(t)$ and $b(t)$:

$$a(t)+w[a(t),t]-t=\eta[a(t),t] \qquad , \qquad -b(t)+w[-b(t),t]-t=\eta[-b(t),t] \tag{8}$$

The formulated coupled problem is nonlinear due to these equations.

Finite element method and coupling

Within the finite element method the beam is divided into N elements and the beam deflection is represented inside each element with the help of four polynomials of the third order $N_{ij}(\xi)$:

$$w(x,t)=\sum_{j=1}^{N}\sum_{i=1}^{4}a_{ji}(t)N_{ij}(\xi) \tag{9}$$

where ξ is the local coordinate within the element j, $-1<\xi<1$, $x=x_j+\ell(\xi+1)/2$, ℓ is the element length, x_1 corresponds to the left edge of the beam and $x_N+\ell$ to its right edge, $x_{j+1}=x_j+\ell$. The unknown coefficients $a_{ji}(t)$ are referred to as principal coordinates of the beam deflection.
Variational form of the Euler beam equation and representation (9) provide system of $4N$ differential equations with respect to the principal coordinates

$$\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}=\mathbf{f}(t), \tag{10}$$

where

$$\mathbf{a}(t)=(a_{11},a_{12},a_{13},a_{14},a_{21},a_{22},.....)^T, \quad \mathbf{f}(t)=(f_{11},f_{12},f_{13},f_{14},f_{21},f_{22},.....)^T, \tag{11}$$

$$f_{ji}(t)=\int_{x_j}^{x_{j+1}}p(x,0,t)N_{ij}(\xi)dx, \qquad \xi=2(x-x_j)/\ell-1 \tag{12}$$

and $p(x,0,y) = -\varphi_t(x,0,t)$.
Note that $p = 0$ outside the contact region, where $x > a(t)$ or $x < -b(t)$. Therefore $f_{ji}(t) \equiv 0$ if j-th element is not wetted, $(x_j, x_{j+1}) \cap (-b(t), a(t)) = \emptyset$.
After some manipulations we obtain:

$$f_{ji}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\ell}{2}\int_{-1}^{1} \varphi(x_j + \ell(\xi+1)/2, 0, t) N_{ij}(\xi) d\xi\right] \tag{13}$$

The coupling procedure is similar to that applied in [2] except that the hydrodynamic coefficients are different. First of all, thanks to the condition (3) and representation (9), we can decompose the potential under the contact region in the following form:

$$\varphi(x,0,t) = \varphi_0(x,a,b) + \sum_{j=1}^{N}\sum_{i=1}^{4} \dot{a}_{ji}(t)\varphi_{ij}(x,a,b), \tag{14}$$

where $\varphi_{ij}(x,a,b) = \phi_{ij}(x,0,a,b)$ with $\phi_{ij}(x,y,a,b)$ being solutions of the boundary value problems:

$$\Delta\phi_{ij} = 0 \qquad (y < 0), \tag{15}$$

$$\phi_{ij} = 0 \qquad (y = 0 \quad x > a \quad \text{or} \quad x < -b), \tag{16}$$

$$\frac{\partial \phi_{ij}}{\partial y} = N_{ij}[2(x - x_j)/\ell - 1] \qquad (y = 0, \quad x \in (x_j, x_{j+1}) \cap (-b, a)), \tag{17}$$

$$\frac{\partial \phi_{ij}}{\partial y} = 0 \qquad (y = 0, \quad x \in (-b, a) \backslash (x_j, x_{j+1})), \tag{18}$$

and $\varphi_0(x,a,b) = \phi_0(x,0,a,b)$ with the function $\phi_0(x,y,a,b)$ being solution of equations (1) - (4) for the rigid wedge case (i.e. $w(x,t) \equiv 0$).
Substituting (14) into (13) and combining the result with (10), we end up with the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\mathbf{M} + \mathbf{S})\dot{\mathbf{a}} + \mathbf{f}_0\right] + \mathbf{K}\mathbf{a} = 0, \tag{19}$$

where $\mathbf{f}_0 = \mathbf{f}_0(a,b)$ is the vector-function with components

$$[\mathbf{f}_0]_{ji} = \frac{\ell}{2}\int_{-1}^{1} \varphi_0(x_j + \ell(\xi+1)/2, a, b) N_{ij}(\xi) d\xi, \tag{20}$$

and $\mathbf{S}$ is the matrix of added masses, $\mathbf{S} = \mathbf{S}(a,b)$, with elements

$$[\mathbf{S}]_{ji}^{nm} = \frac{\ell}{2}\int_{-1}^{1} \varphi_{mn}(x_j + \ell(\xi+1)/2, a, b) N_{ij}(\xi) d\xi. \tag{21}$$

It is convenient to introduce new unknown vector $\mathbf{d} = (\mathbf{M} + \mathbf{S})\dot{\mathbf{a}} + \mathbf{f}_0$, with the help of which the original coupled problem is reduced to the system of ordinary differential equations

$$\dot{\mathbf{d}} = -\mathbf{K}\mathbf{a}, \tag{22}$$

$$\dot{\mathbf{a}} = (\mathbf{M} + \mathbf{S}(a,b))^{-1}[\mathbf{d} - \mathbf{f}_0(a,b)]. \tag{23}$$

The initial conditions are $\mathbf{a}(0) = 0, \mathbf{d}(0) = 0$.
It is important to note that the Wagner condition (8) should also be reformulated according to the finite element representation (9), in order to properly close the problem. The details are not given here and can be found in [3].

Hydrodynamic coefficients

The main difficulty of the analysis is the evaluation of the added mass matrix $\mathbf{S}$, excitation vector $\mathbf{f}_0$ and the wetted part of the body $a(t), b(t)$. Here we concentrate on the evaluation of the added mass.
We rewrite equation (21) using the global coordinate x instead of the local variable ξ

$$[\mathbf{S}]_{ji}^{nm}(a,b) = \int_{x_j}^{x_{j+1}} \varphi_{mn}(x,a,b) N_{ij}(2[x - x_j]/\ell - 1) dx, \tag{24}$$

where $1 \leq n, j \leq N$, $m, i = 1,2,3,4$. It should be noted once again that in (24) only the numbers j and n, for which both intervals $[x_j, x_{j+1}]$ and $[x_n, x_{n+1}]$ have non-empty intersections with the contact region $[-b, a]$, are considered. For all other numbers j and n the integral in (24) is equal to zero.
It is convenient to introduce the new function

$$\hat{N}_{ij}(x) = \begin{cases} 0, & x > x_{j+1}, \quad x < x_j \\ N_{ij}[2[x - x_j]/\ell - 1] & x_j < x < x_{j+1}, \end{cases} \tag{25}$$

with the help of which integral (24) takes the form

$$[\mathbf{S}]_{ji}^{nm}(a,b) = \int_{-\infty}^{\infty} \varphi_{mn}(x,a,b)\hat{N}_{ij}(x)\mathrm{d}x = \int_{-b}^{a} \varphi_{mn}(x,a,b)\hat{N}_{ij}(x)\mathrm{d}x. \tag{26}$$

Denoting $\tilde{N}_{ij}(x) = \int_{-b}^{x} \hat{N}_{ij}(x_0)\mathrm{d}x_0$, integral (24) can be written as

$$[\mathbf{S}]_{ji}^{nm}(a,b) = -\int_{-b}^{a} \tilde{N}_{ij}(x)\frac{\partial \phi_{mn}}{\partial x}(x,0,a,b)\mathrm{d}x. \tag{27}$$

After introducing the new variable λ, so that $x = x(\lambda) = \frac{a+b}{2}\lambda + \frac{a-b}{2}$ we can write:

$$[\mathbf{S}]_{ji}^{nm}(a,b) = -\frac{a+b}{2}\int_{-1}^{1} \tilde{N}_{ij}[x(\lambda)]\frac{\partial \phi_{mn}}{\partial x}\left[\frac{a+b}{2}\lambda + \frac{a-b}{2}, 0, a, b\right]\mathrm{d}\lambda. \tag{28}$$

A solution of the problem (15)–(18) for $\partial\phi_{mn}/\partial x$ can be found under the form

$$\frac{\partial \phi_{mn}}{\partial x}\left(\frac{a+b}{2}\lambda + \frac{a-b}{2}, 0\right) = \frac{1}{\pi\sqrt{1-\lambda^2}}\mathbf{P.v.}\int_{-1}^{1} \frac{\hat{N}_{mn}[x(\tau)]\sqrt{1-\tau^2}\mathrm{d}\tau}{\tau - \lambda}. \tag{29}$$

Substituting (29) into (28) and rearranging the result, we obtain that all elements of the added mass matrix, as well as the forcing vector $\mathbf{f}_0$ and the wetting correction, can be evaluated analytically [3] which makes the procedure very efficient.

Results and discussion

In figure 2, we show the results for the non-dimensional deflection, and its second derivative, of the wedge right plating during impact, obtained by the present method and normal mode method [1]. The plate is freely supported at its edges, the half-length L equals $0.5m$, the beam thickness is $1cm$, its density $7850Kg/m^3$, deadrise angle γ is 10 degrees, and the velocity equals $4m/s$. The time corresponds to the instant where the plate is wet at 50%. We can see very good agreement between two methods.

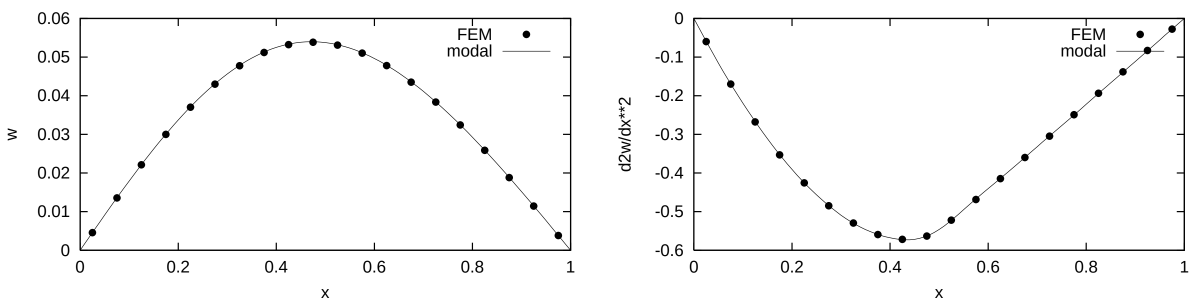

Figure 2: Plate deflection (left) and its second derivative (right) obtained by modal and FEM approach.

References

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