Uniqueness in the water-wave problem for bodies intersecting the free surface at arbitrary angles

Nikolay KUZNETSOV

Laboratory for Mathematical Modelling of Wave Phenomena, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol'shoy pr. 61, St. Petersburg 199178, Russian Federation,

1 Inroduction

In 1950, John [1] proved the first result on uniqueness in the water-wave problem (WWP for short, in what follows). He considered surface-piercing bodies subjected to the condition now usually referred to as John's condition. In the two-dimensional case it includes the following two requirements:

(i) there is one and only one surface-piercing cylinder in the set of cylinders forming the immersed obstacle between the horizontal free surface and the rigid bottom;

(ii) the whole obstacle is contained within the strip between two vertical lines through the points, where the surface-piercing contour intersects the free surface of water, outside of this strip the bottom is horizontal.

Much later, Simon and Ursell [2] demonstrated that if condition (i) holds, then condition (ii) can be replaced by a weaker one. Namely, if the depth is infinite, then the whole obstacle must be confined to the angular domain between the lines inclined at $\pi/4$ to the vertical and going through two points, where the surface-piercing contour intersects the free surface. If the depth is finite, then the result in [2] requires that the whole obstacle is confined to an angular domain between the lines going through the same two points and inclined at a certain angle to the vertical that is smaller than $\pi/4$.

Further development of the linear theory of water waves (see the book [3] for an updated account of this theory) shown that condition (i) is essential for the uniqueness theorem to be true. This became clear when McIver [4] constructed an example of a nontrivial solution to the homogeneous WWP with two surface-piercing cylinders each satisfying modified (ii), but separated by a nonzero spacing. McIver's result has increased interest in determining conditions under which a solution is unique, but even before that, the importance of this question was emphasized by Ursell, who placed it first in his list [5] of unsolved and unfinished problems in the theory of water waves. In the passed few years, some progress was achieved concerning the uniqueness in the WWP in two and three dimensions. However, conditions, providing uniqueness for bodies which intersect the free surface at angles to the vertical greater than those obtained in [2], were not found so far. Our aim is to fill in this gap, at least partially. We will be mainly concerned with the case of a surface-piercing cylinder in water of infinite depth, but results for other geometries of obstacles will be also mentioned.

2 Two equivalent statements of the problem

Let W denote the cross-section of a domain occupied by an inviscid, incompressible, heavy fluid (water). It is assumed that $W = I\!R^2_- \setminus \overline{B}$, where $I\!R^2_- = \{-\infty < x < +\infty, y < 0\}$ and B is a simply connected domain in $I\!R^2_-$ attached to $\partial I\!R^2_-$ and such that: (a) for $S = \partial B \cap I\!R^2_-$, the closure $\overline{S}$ is a piecewise C^2-curve without cusps; (b) $\overline{S}$ is not tangent to the x-axis at $(\pm a, 0)$ which, without loss of generality, are taken as the endpoints of $\overline{S}$. Thus $\overline{B}$ is the cross-section of the immersed part of an infinitely long cylinder floating in the water surface. If the surface tension is negligible and the water motion is assumed to be irrotational, then the small-amplitude oscillations of water such that the radian frequency is equal to ω are described mathematically by a complex-valued velocity potential ϕ satisfying the following boundary value problem:

$$\nabla^2\phi = 0 \ \text{ in } W, \quad \phi_y = \nu\phi \ \text{ on } F = \{|x| > a,\, y = 0\}, \quad \partial\phi/\partial n = f \ \text{ on } S, \tag{1}$$

$$\phi_x \mp i\nu\phi = o(1) \ \text{ uniformly in } y \in (-\infty, 0] \text{ as } \pm x \to +\infty. \tag{2}$$

This problem must be complemented by the condition that the Dirichlet integral of ϕ is locally finite. In (1), f is a given function on S that depends on the type of problem (radiation or scattering) and ν is a nonnegative spectral parameter equal to ω^2/g, where g is the acceleration due to gravity.

Since we are interested in the question of uniqueness, we put $f = 0$ in which case the following conditions hold (cf. [3], Section 2.2): $\phi(x, y) = O\left([x^2 + y^2]^{-1/2}\right)$ and $|\nabla\phi| = O\left([x^2 + y^2]^{-1}\right)$ as $x^2 + y^2 \to \infty$. They imply that

$$\int_W |\nabla\phi|^2 \,\mathrm{d}x\mathrm{d}y < \infty, \qquad \int_F \phi^2 \,\mathrm{d}x < \infty, \tag{3}$$

whose meaning is that the kinetic and potential energy of waves per unit length of cylinder's generators is finite. If the Neumann condition is homogeneous, then conditions (3) can be imposed instead of (2), and so ϕ can be assumed to be real because if it were complex, then both the real and imaginary parts would separately satisfy the problem.

Let us rewrite problem (1), (3) using the bipolar coordinates (u,v) because this leads to essential simplifications in the proof of uniqueness. The standard way to introduce them is as follows (see, for example, [6], Section 10.1):

$$x = a\sinh u/(\cosh u - \cos v), \quad y = a \sin v/(\cosh u - \cos v), \tag{4}$$

where a was defined in condition (b) for S. Let us describe some properties of (4) that maps conformally $\mathbb{R}^2_-$ onto $\{-\infty < u < +\infty,\ -\pi < v < 0\}$. The metric coefficients are $g_{11} = g_{22} = a^2/(\cosh u - \cos v)^2$. The points $(\pm a, 0)$ on the (x,y) plane go to infinity on the (u,v) plane, the origin on the latter plane corresponds to the point at infinity on the (x,y) plane, and each coordinate line

$$\{\pm u > 0,\ v = \sigma\}, \quad \sigma = \text{const}, \quad -\pi < \sigma < 0, \tag{5}$$

is represented on the (x,y) plane by the following circular arc:

$$x^2 + (y - a\cot\sigma)^2 = a^2(\cot^2\sigma + 1), \quad \pm x > 0, \quad y < 0, \tag{6}$$

that has $(\pm a, 0)$ as one of its endpoints (see Fig. 1 below, where several such arcs are shown by dashed lines; the second family of the coordinate lines in $\mathbb{R}^2_-$, semicircles which have their centres on the x-axis and are orthogonal to curves (6), will not be used in the present context). Moreover, $\{-\infty < u < +\infty,\ v = -\pi\}$ and $\{\pm u > 0,\ v = 0\}$ are the images of $\{|x| < a,\ y = 0\}$ and $\{\pm x > a,\ y = 0\}$, respectively. Therefore, F is mapped onto the whole u-axis (this is the first reason for using the bipolar coordinates) and the image of W is a curvilinear strip denoted by $\mathcal{W}$. Apart from the u-axis, the boundary $\partial\mathcal{W}$ includes the image of S. It follows from properties of (4) that this curve denoted by $\mathcal{S}$ lies between the u-axis and the horizontal line $v = -\pi$ and asymptotes the horizontal lines $v = -\alpha_\pm$ as $u \to \pm\infty$ (this is the second reason for introducing (u,v)). Here $\alpha_\pm \in (0,\pi)$ is the angle between S and F at $(\pm a, 0)$.

Let $\varphi(u,v) = \phi(x(u,v), y(u,v))$, where $x(u,v)$ and $y(u,v)$ are given by (4). Then we get from (1) with the homogeneous Neumann condition:

$$\nabla^2\varphi = 0 \quad \text{in } \mathcal{W}, \qquad (\cosh u - 1)\varphi_v = \nu a\,\varphi \quad \text{for } v = 0, \qquad \nabla\varphi\cdot\mathbf{n} = 0 \quad \text{on } \mathcal{S}. \tag{7}$$

Here $\mathbf{n}$ is the unit normal to $\mathcal{S}$ exterior with respect to $\mathcal{W}$. Moreover, conditions (3) imply that

$$\int_{\mathcal{W}} |\nabla\varphi|^2 \,\mathrm{d}u\mathrm{d}v < \infty, \qquad \int_{-\infty}^{+\infty} \frac{\varphi^2(u,0)}{\cosh u - 1}\,\mathrm{d}u < \infty, \tag{8}$$

which completes the second statement of the homogeneous WWP.

3 Integral identities

According to (8), the first Green's identity can be applied in $\mathcal{W}$, and in view of (7) we get:

$$\int_{\mathcal{W}} |\nabla\varphi|^2 \,\mathrm{d}u\mathrm{d}v = \nu a \int_{-\infty}^{+\infty} \frac{\varphi^2(u,0)}{\cosh u - 1}\,\mathrm{d}u. \tag{9}$$

Another integral identity will be derived from the following identity proposed by Vainberg and Maz'ya (see [3], Subsection 2.2.2): $(2u\varphi_u + \varphi)\nabla^2\varphi = \nabla\cdot(2u\varphi_u + \varphi)\nabla\varphi - 2\varphi_u^2 - \left(u|\nabla\varphi|^2\right)_u$, which is straightforward to verify. Let us integrate this identity over $\mathcal{W}' = \mathcal{W} \cap \{|u| < b\}$, where b is sufficiently large. Using (7) and the divergence theorem, we get

$$2\int_{\mathcal{W}'} \varphi_u^2 \,\mathrm{d}u\mathrm{d}v + \int_{\mathcal{S}'} \mathbf{u}\cdot\mathbf{n}\,|\nabla\varphi|^2 \,\mathrm{d}S = \int_{-b}^{+b} \left[2u\varphi_u(u,0) + \varphi(u,0)\right]\varphi_v(u,0)\,\mathrm{d}u + \sum_\pm \pm \int_{\mathcal{C}_\pm} (2u\varphi_u + \varphi)\,\varphi_u\,\mathrm{d}v, \tag{10}$$

where $\mathcal{S}' = \mathcal{S} \cap \{|u| < b\}$, $\mathbf{u} = (u, 0)$, $\sum_{\pm}$ denotes the summation of two terms corresponding to the upper and lower signs, respectively, and $\mathcal{C}_{\pm} = \mathcal{W}' \cap \{u = \pm b\}$. Using the free-surface boundary condition, we transform the first term in the right-hand side as follows:

$$\nu a \int_{-b}^{+b} [2u\varphi_u(u,0) + \varphi(u,0)] \, \frac{\varphi(u,0)}{\cosh u - 1} \, \mathrm{d}u = \nu a \int_{-b}^{+b} \frac{u \sinh u}{(\cosh u - 1)^2} \, \varphi^2(u,0) \, \mathrm{d}u + \nu a \left[\frac{u \, \varphi^2(u,0)}{\cosh u - 1} \right]_{u=-b}^{u=+b}.$$

Here the last expression is obtained by integration by parts. From (3) it follows that $\phi(x,y)$ tends to constants as $(x,y) \to (\pm a, 0)$, and so $\varphi(u,v)$ has the same property as $u \to \pm\infty$. Therefore, the integrated term in the last equality tends to zero as $b \to \infty$. The integral in the right-hand side of the last equality converges as $b \to \infty$ in view of (8). Furthermore, (8) implies that there exists a sequence $\{b_k\}_1^\infty$ such that the last sum in (10) tends to zero as $b_k \to \infty$. Passing to the limit in the transformed equation (10) with $b = b_k$ and $k \to \infty$, we arrive at the following integral identity:

$$2 \int_{\mathcal{W}} \varphi_u^2 \, \mathrm{d}u\mathrm{d}v + \int_{\mathcal{S}} \mathbf{u} \cdot \mathbf{n} \, |\nabla \varphi|^2 \, \mathrm{d}S = \nu a \int_{-\infty}^{+\infty} \frac{u \sinh u}{(\cosh u - 1)^2} \, \varphi^2(u,0) \, \mathrm{d}u.$$

Subtracting this from (9) multiplied by two, we get

$$2 \int_{\mathcal{W}} \varphi_v^2 \, \mathrm{d}u\mathrm{d}v - \int_{\mathcal{S}} \mathbf{u} \cdot \mathbf{n} \, |\nabla \varphi|^2 \, \mathrm{d}S + \nu a \int_{-\infty}^{+\infty} \frac{u \sinh u - 2(\cosh u - 1)}{(\cosh u - 1)^2} \, \varphi^2(u,0) \, \mathrm{d}u = 0. \tag{11}$$

This identity will be used in the next section for proving the uniqueness theorem for the WWP.

4 Conditions of uniqueness

It is easy to see that $u \sinh u - 2(\cosh u - 1) \geq 0$ for all $u \in I\!R$ and the equality holds only for $u = 0$. If $\mathbf{u} \cdot \mathbf{n} \leq 0$ on $\mathcal{S}$, then the expression in the left-hand side in (11) is strictly positive for a nontrivial φ. Therefore, (11) leads to a contradiction unless φ is equal to zero identically. Hence, the following assertion is proved.

Proposition 1. *Let $\mathbf{u} \cdot \mathbf{n} \leq 0$ on $\mathcal{S}$, then the homogeneous boundary value problem* (7), (8) *has only a trivial solution for all positive values of ν.*

Let us reformulate this uniqueness theorem in terms of the WWP. For this purpose we note that the inequality $\mathbf{u} \cdot \mathbf{n} \leq 0$ that must hold on $\mathcal{S}$ means that each horizontal half-axis (5) that begins on the v-axis and goes to $\pm\infty$, crosses $\mathcal{S}$ at most once. Moreover, *all transversal intersections of these half-axes with $\mathcal{S}$ are points of entry into $\mathcal{W}$*. Since the circular arcs (6) beginning on the negative y-axis and ending at $(\pm a, 0)$ correspond to the half-axes (5), we get the following equivalent formulation of the uniqueness theorem proven.

Proposition 2. *Let all transversal intersections of curves* (6) *with S be points of entry into W, then the homogeneous* WWP *has only a trivial solution for all positive values of ν.*

5 Discussion

Uniqueness has been established in the two-dimensional WWP for all values of the radian frequency provided that the contour S, bounding the cross-section of a surface-piercing cylinder, satisfies the transversality condition with the circular arcs (6) which belong to one of two families of the coordinate lines of the bipolar coordinates (4). This condition requires that curves (6) enter into the water domain; several similar conditions of uniqueness were obtained earlier in a number of papers by Vainberg, Maz'ya, and Kuznetsov (see also [3], Chapters 2 and 3). The novelty of the present result is that: **1**) it imposes no restriction on the angles formed by S with the free surface F; **2**) S is not supposed to be symmetric about a vertical axis. The second feature distinguishes the present result from that of McIver [7]. She also considered the two-dimensional WWP and proved a condition of uniqueness that has property **1**, but her result has a drawback being valid only for symmetric solutions (they are given by even in x functions) when the obstacle is symmetric about the y-axis.

It is easy to give examples of contours S ($\mathcal{S}$) for which Proposition 2 (Proposition 1) is true. It is obvious that Proposition 2 is valid for every circular cylinder belonging to the family (6). Another simple example is a cylinder whose cross-section is an ellipse with the following properties. It goes through $(\pm a, 0)$ and its major axis is vertical.

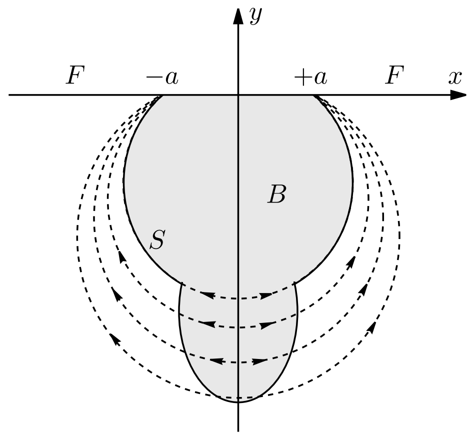

Figure 1: An example of B for which Proposition 2 holds; dashed lines are curves (6). Both angles between S and F are equal to each other and are smaller than $\pi/4$.

If $\mathcal{S}$ is given by

$$v = -\alpha + (\beta + u^2)^{-1}, \quad \text{where} \quad 0 < \alpha < \pi \quad \text{and} \quad 0 < \beta^{-1} < \alpha,$$

then Proposition 1 obviously holds. In order to describe the corresponding curve S one has to substitute the latter expression for v into (4), thus obtaining parametric equations of S with u as the parameter varying on $\mathbb{R}$. Both angles between F and the curve S, obtained in this way, are equal to α.

If S is composed of parts for either of which all transversal intersections with curves (6) are points of entry of curves (6) into the water domain, then Proposition 2 is true for this curve S. For example, a part of an ellipse with the vertical major axis can be attached from below to a circular segment given by (6) (see Fig. 1, where all curves (6) shown by dashed lines enter W except for that one whose continuations are the circular parts of S given by (6)).

Furthermore, the water domain W may have finite depth because Proposition 2 remains true if an infinite curve H that bounds W from below has the following two properties: (I) H asymptotes a horizontal line at infinity; (II) all transversal intersections of H with curves (6) are points of entry of curves (6) into W. In particular, (II) holds when H is a horizontal straight line. The obstacle immersed in water may also include totally submerged cylinders in addition to the surface-piercing one. Of course, the boundary of either totally submerged body must satisfy the same transversality condition with curves (6).

References

[1] F. John, On the motion of floating bodies. II, *Comm. Pure Appl. Math.* **3** (1950) 45–101.

[2] M. J. Simon, F. Ursell, Uniqueness in linearized two-dimensional water-wave problems, *J. Fluid Mech.* **148** (1984) 137–154.

[3] N. Kuznetsov, V. Maz'ya, B. Vainberg, *Linear Water Waves: A Mathematical Approach*, Cambridge University Press, 2002.

[4] M. McIver, An example of non-uniqueness in the two-dimensional linear water-wave problem, *J. Fluid Mech.* **315** (1996) 257–266.

[5] F. Ursell, Some unsolved and unfinished problems in the theory of waves, In *Wave Asymptotics*, Cambridge University Press, 1992.

[6] P.M. Morse, H. Feshbach, *Methods of Theoretical Physics*, Part II, McGraw-Hill, 1953.

[7] M. McIver, Uniqueness and trapped modes for a symmetric structure, In *Proc. of the 14th Int. Workshop on Water Waves and Floating Bodies*, University of Michigan, Ann Arbor, 1999.

Discusser: D.V. Evans

You make a transformation using bipolar coordinates and then use as equality due to Maz'ya to establish an integral identity which leads to a simple geometrical condition for uniqueness.Is it possible to obtain the result more simply using John's ideas on integration by parts, to the transformed equations without having to make use of the Maz'ya identity?

Author's reply:

The problem arising after conformal mapping (this is equivalent to using bipolar coordinates) contains a variable coefficient in the free-surface boundary condition. Therefore, it is not simple at all to use the John's technique in this case instead of the Maz'ya-Vainberg identity. Neverthless, it is worth to try to apply John's method, and I appreciate your remark as a suggestion for future work.