Generalized Potential-Flow Representations

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Classical potential-flow representation

The potential $\widetilde{\phi} = \phi(\widetilde{\mathbf{x}})$ at a field point $\widetilde{\mathbf{x}}$ within a flow domain bounded by a closed boundary surface Σ is defined in terms of the boundary values of the potential ϕ and its normal derivative $\mathbf{n}\cdot\nabla\phi$ by the classical boundary-integral representation

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,(\,G\,\nabla\phi - \phi\,\nabla G\,)\cdot\mathbf{n} \tag{1}$$

$d\mathcal{A}$ and $\mathbf{n} = (\,n^x, n^y, n^z\,)$ stand for a differential element of area and a unit vector normal to the boundary surface Σ ($\mathbf{n}$ points into the flow domain), respectively, at a point $\mathbf{x}$ of Σ, and $\nabla = (\partial_x\,, \partial_y\,, \partial_z)$ represents the gradient with respect to $\mathbf{x}$. The representation (1) defines the potential in terms of boundary distributions of sources (with strength $\mathbf{n}\cdot\nabla\phi$) and normal dipoles (strength ϕ), and involves a Green function G and the first derivatives of G. The potential representation (1) holds for a field point $\widetilde{\mathbf{x}}$ inside the flow domain, strictly outside the boundary surface Σ. This restriction stems from the well-known property that the potential defined by the dipole distribution in (1) is not continuous at the surface Σ. Indeed, $\widetilde{\phi}$ on the left of (1) becomes $\widetilde{\phi}/2$ at a point $\widetilde{\mathbf{x}}$ of the boundary surface Σ (if Σ is smooth at $\widetilde{\mathbf{x}}$).

Weakly-singular potential-flow representation

An alternative boundary-integral representation is given in [1]. This representation is obtained using a vector Green function $\mathbf{G}$ associated with the scalar Green function G in (1) via the relation

$$\nabla\times\mathbf{G} = \nabla G \tag{2}$$

The relation (2) implies that $\mathbf{G}$ is no more singular than G. The vector Green function

$$\mathbf{G} = (G_y^z\,, -G_x^z\,, 0) \tag{3}$$

is considered here, as in [1]. A subscript or superscript attached to G indicates differentiation or integration, respectively. The identity $\nabla\times(\,\phi\,\mathbf{G}\,) = \phi\,\nabla\times\mathbf{G} + \nabla\phi\times\mathbf{G}$ and (2) yield

$$[\,\nabla\times(\,\phi\,\mathbf{G}\,)\,]\cdot\mathbf{n} = \phi\,\nabla G\cdot\mathbf{n} + (\,\nabla\phi\times\mathbf{G}\,)\cdot\mathbf{n}$$

Integration of this identity over a closed boundary surface Σ then yields

$$\int_\Sigma d\mathcal{A}\;\phi\,\nabla G\cdot\mathbf{n} = \int_\Sigma d\mathcal{A}\,(\mathbf{G}\times\nabla\phi)\cdot\mathbf{n} \tag{4}$$

The field point $\widetilde{\mathbf{x}}$ in (4) is inside the flow domain, strictly outside the boundary surface Σ. The transformation (4) expresses a surface integral involving the potential ϕ and the derivative ∇G of a Green function G as an integral that involves $\nabla\phi$ and the vector Green function $\mathbf{G}$, which is comparable to G. Thus, the transformation (4) corresponds to an integration by parts $(\,\phi\,,\nabla G) \longrightarrow (\nabla\phi, G)$.

Substitution of the transformation (4) into the classical potential representation (1) yields

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,(G\,\nabla\phi - \mathbf{G}\times\nabla\phi)\cdot\mathbf{n} \tag{5}$$

This alternative potential representation only involves a Green function G and the related vector Green function $\mathbf{G}$, which is no more singular than G as already noted. Thus the potential representation (5) is weakly singular in comparison to the classical representation (1), which involves G and ∇G. The potential $\widetilde{\phi}$ defined by the weakly-singular potential representation (5) is continuous at the boundary surface Σ, whereas the potential $\widetilde{\phi}$ defined by the classical boundary-integral representation (1) is not.

Generalized potential-flow representation

The weakly-singular boundary-integral representation (5) and the classical representation (1) can be regarded as special cases of a more general family of potential representations, as now shown. The basic potential representation (1) can be expressed as

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,[\,G\,\nabla\phi - P\phi\,\nabla G - (1-P)\,\phi\,\nabla G\,]\cdot\mathbf{n} \tag{6a}$$

where $P = P(\mathbf{x}\,;\widetilde{\mathbf{x}})$ stands for a function of $\mathbf{x}$ and $\widetilde{\mathbf{x}}$. Expression (4), with ϕ replaced by $P\phi$, yields

$$\int_\Sigma d\mathcal{A}\, P\phi\nabla G\cdot\mathbf{n} = \int_\Sigma d\mathcal{A}\,[\,\mathbf{G}\times\nabla(P\phi)]\cdot\mathbf{n} \tag{6b}$$

The potential representation (6a) and the transformation (6b) yield

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,[\,G\,\nabla\phi - \mathbf{G}\times\nabla(P\phi) - (1-P)\,\phi\,\nabla G\,]\cdot\mathbf{n} \tag{7}$$

The potential representation (7) generalizes the classical representation (1) and the weakly-singular representation (5), which correspond to the special cases $P = 0$ and $P = 1$, respectively.

Potential representation for free-space Green function

The potential representation (7) is now considered for the simplest case when the Green function is chosen as the fundamental free-space Green function, given by $4\pi G = -1/r$ with

$$r = \sqrt{\mathbf{X}\cdot\mathbf{X}} \qquad \mathbf{X} = (\,X,Y,Z\,) = (\,\widetilde{x}-x\,,\widetilde{y}-y\,,\widetilde{z}-z\,) \tag{8}$$

Expressions (7) and (8) yield

$$\widetilde{\phi} = \frac{-1}{4\pi}\int_\Sigma d\mathcal{A}\left(\frac{\nabla\phi}{r} - \mathbf{S}\times\nabla(P\phi) - \phi\,\frac{1-P}{r^2}\,\frac{\mathbf{X}}{r}\right)\cdot\mathbf{n} \tag{9}$$

where $\mathbf{S}$ satisfies the relation $\nabla\times\mathbf{S} = \nabla(1/r)$ and is chosen as $\mathbf{S} = [\,(1/r)^z_y\,, -(1/r)^z_x\,, 0\,]$ in accordance with (3). The function $(1/r)^z$ and its derivatives with respect to x and y are given by

$$(1/r)^z = -\,\mathrm{sign}(Z)\,\ln(\,r + |\,Z\,|\,) \qquad \begin{Bmatrix}(1/r)^z_x\\(1/r)^z_y\end{Bmatrix} = \frac{\mathrm{sign}(Z)}{r+|\,Z\,|}\begin{Bmatrix}X/r\\Y/r\end{Bmatrix}$$

Thus, we have $\mathbf{S} = \mathbf{s}/r$ where $\mathbf{s}$ is given by

$$\mathbf{s} = \frac{\mathrm{sign}(d^z)}{1+|d^z|}(\,d^y, -d^x, 0\,) \quad \text{with} \quad (\,d^x, d^y, d^z\,) = \mathbf{d} = \frac{(\,x-\widetilde{x}\,,y-\widetilde{y}\,,z-\widetilde{z}\,)}{r} \tag{10}$$

This definition of $\mathbf{d}$ yields $\mathbf{d} = -\mathbf{X}/r$, $\nabla r = \mathbf{d}$ and $\nabla(1/r) = -\,\mathbf{d}/r^2$. Thus, (9) becomes

$$\widetilde{\phi} = \frac{-1}{4\pi}\int_{\Sigma_B}\frac{d\mathcal{A}}{r}\left(\nabla\phi - \mathbf{s}\times\nabla(P\phi) + \phi\,\frac{1-P}{r}\,\mathbf{d}\right) \tag{11}$$

In the special cases $P = 0$ and $P = 1$, the potential representation (11) becomes

$$\widetilde{\phi} = \frac{-1}{4\pi}\int_\Sigma\frac{d\mathcal{A}}{r}\left(\nabla\phi + \frac{\phi}{r}\,\mathbf{d}\right)\cdot\mathbf{n} \qquad \text{for } P = 0 \tag{12a}$$

$$\widetilde{\phi} = \frac{-1}{4\pi}\int_\Sigma\frac{d\mathcal{A}}{r}\,(\,\nabla\phi - \mathbf{s}\times\nabla\phi\,)\cdot\mathbf{n} \qquad \text{for } P = 1 \tag{12b}$$

The dipole term in the classical potential representation (12a) is $O(1/r^2)$; this term decays rapidly in the farfield but is strongly singular in the nearfield. The corresponding term in the alternative potential representation (12b) is $O(1/r)$; this term is weakly singular in the nearfield but decays slowly in the farfield. Thus, the alternative potential representations (12a) and (12b) are best suited in the farfield and the nearfield, respectively, and—in that sense—are complementary.

If the function P in (11) vanishes in the farfield and tends to 1 in the nearfield sufficiently rapidly, the integrand of (11) is asymptotically equivalent to the integrands of (12a) and (12b) in the farfield and nearfield limits. E.g., consider the function

$$P = 1/(1+r^2/\ell^2) \qquad \text{where } \ell \text{ corresponds to a transition length scale} \tag{13}$$

Expressions (11) and (13) yield

$$\widetilde{\phi} = \frac{-1}{4\pi}\int_\Sigma\frac{d\mathcal{A}}{r}\left[\nabla\phi - \frac{\mathbf{s}\times\nabla\phi}{1+r^2/\ell^2} + \frac{r\,\phi}{\ell^2+r^2}\left(\mathbf{d} + \frac{2\,\mathbf{s}\times\mathbf{d}}{1+r^2/\ell^2}\right)\right]\cdot\mathbf{n} \tag{14}$$

where $\mathbf{s}$ and $\mathbf{d}$ are given by (10). The potential representation (14) is identical to the representations corresponding to $P=0$ and $P=1$ in the limits $\ell=0$ and $\ell=\infty$, respectively. The integrand of (14) is identical to the integrands corresponding to $P=0$ and $P=1$ in the farfield and nearfield limits $r/\ell\to\infty$ and $r/\ell\to 0$, respectively.

Application to 3D wave diffraction-radiation with forward speed

For potential flows in deep water, the boundary surface Σ in the alternative flow representations (1), (5), (7) becomes $\Sigma=\Sigma_B\cup\Sigma_0$, where Σ_B stands for the hull of a body (ship or offshore structure), or a surface that encloses a body, and Σ_0 is the portion of the mean free surface (taken as the plane $z=0$) outside Σ_B if Σ_B pierces the free surface. The Green function in (1), (5), (7) is commonly chosen as the basic free-space Green function $1/r$ (Rankine-source method) or as a Green function that satisfies the linearized free-surface boundary condition (free-surface Green-function method). For wave diffraction-radiation by a ship advancing at constant speed $\mathcal{U}$ in time-harmonic waves (frequency ω), two alternative free-surface Green functions are given in [2] :

$$4\pi G=G^R+\begin{Bmatrix}G^F\\ i\,G^W\end{Bmatrix}\qquad\text{with}\qquad G^R=\frac{-1}{r}+\frac{1}{r_*}-\frac{2}{r_F}+\frac{2}{r_{Ff}}\tag{15a}$$

$$G^F=\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{-\infty}^{\infty}d\beta\int_{-\infty}^{\infty}d\alpha\;\frac{A\,e^{Z_*\,k-i\,(\alpha X+\beta Y)}}{D+i\,\varepsilon\,D_1}\qquad G^W=W^+-W^--W^i\tag{15b}$$

The Green function G^R+G^F satisfies the linear free-surface boundary condition everywhere, i.e. in both the farfield (where the linear free-surface condition is valid) and the nearfield (where the linear condition is only an approximation, because of nearfield effects). The Green function $G^R+i\,G^W$ satisfies the linear free-surface condition accurately in the farfield but only approximately in the nearfield. The Rankine component G^R in (15a) is given by four elementary Rankine sources, defined in [2], that account for the dominant terms in both the nearfield and farfield asymptotic approximations to the non-oscillatory local-flow component contained in the Green function associated with wave diffraction-radiation with forward speed (and the special cases corresponding to $\mathcal{U}=0$ or $\omega=0$). The components $W^{\pm}$ and W^i in (15b) represent distinct wave systems generated by a pulsating source advancing at constant speed. These wave components are defined by *one*-dimensional Fourier superpositions of elementary waves. The integrands of the Fourier integrals that define these wave components are continuous and only involve trigonometric and hyperbolic functions of real arguments, and the limits of integration for these Fourier integrals are independent of the coordinates $(X,Y,Z_*)=(\widetilde{x}-x\,,\widetilde{y}-y\,,\widetilde{z}+z)$. Thus, the wave component G^W is considerably simpler than the Fourier component G^F, defined by a *singular double* Fourier integral. The one-dimensional Fourier integrals that define the wave components $W^{\pm}$ and W^i in (15b) are given in [2]. The dispersion functions D and D_1 and the amplitude function A in the Fourier component G^F are given in (20) with $F=\mathcal{U}/\sqrt{gL}$ and $f=\omega\sqrt{L/g}$. Furthermore, $k=\sqrt{\alpha^2+\beta^2}$.

Substitution of the decompositions G^R+G^F or $G^R+i\,G^W$ of the Green function into the potential representation (7) yields the alternative representations

$$4\pi\widetilde{\phi}=\widetilde{\phi}^R+\widetilde{\phi}^F\qquad\qquad 4\pi\widetilde{\phi}=\widetilde{\phi}^R+i\,\widetilde{\phi}^W\tag{16}$$

The potentials $\widetilde{\phi}^R,\widetilde{\phi}^F,\widetilde{\phi}^W$ are defined by the basic potential representation (1), with G taken as G^R,G^F,G^W. These basic representations of the potential $\widetilde{\phi}^R,\widetilde{\phi}^F,\widetilde{\phi}^W$ can be modified using the transformation (6b), in the manner shown in (7). The three components G^R,G^F and G^W in (15) satisfy the Laplace equation. The transformation (6b) can therefore be applied separately to the potentials $\widetilde{\phi}^R,\widetilde{\phi}^F$ and $\widetilde{\phi}^W$. Thus, the Rankine component $\widetilde{\phi}^R$ can be expressed as

$$\begin{aligned}\widetilde{\phi}^R=&\int_{\Sigma_B}d\mathcal{A}\,[\,G^R\,\nabla\phi-\mathbf{G^R}\times\nabla(P\phi)-(1-P)\,\phi\,\nabla G^R\,]\cdot\mathbf{n}\\ &+\int_{\Sigma_0}dx\,dy\,[\,(P\phi)_x\,(G^R)^z_x+(P\phi)_y\,(G^R)^z_y+(1-P)\,\phi\,G^R_z-\phi_z\,G^R\,]\end{aligned}\tag{17}$$

where $\mathbf{G^R}=[\,(G^R)^z_y\,,-(G^R)^z_x\,,0\,]$ in accordance with (3).

The integrand of the integral over the free surface Σ_0 in (17) can be expressed as

$$(P\phi)_x\,(\pi^R)^{zz}_x+(P\phi)_y\,(\pi^R)^{zz}_y+(1-P)\,\phi\,\pi^R-G^R\,\pi^\phi+f^2a^f+i\,\widehat{\tau}\,a^\tau+F^2a^F$$

where $\widehat{\tau} = 2fF$ and $\pi^R, \pi^\phi, a^f, a^\tau, a^F$ are defined as

$$\pi^R = G^R_z + F^2 G^R_{xx} - f^2 G^R - i\widehat{\tau}\, G^R_x \qquad \pi^\phi = \phi_z + F^2 \phi_{xx} - f^2\phi + i\widehat{\tau}\,\phi_x \tag{18}$$
$$a^f = [P\phi\,(G^R)^{zz}_x]_x + [P\phi\,(G^R)^{zz}_y]_y$$
$$a^\tau = [(P\phi)_y\,(G^R)^{zz}_y]_x - [(P\phi)_x\,(G^R)^{zz}_y]_y + [(1-P)\,\phi\, G^R]_x$$
$$a^F = (\phi_x\, G^R)_x - [(P\phi)_y\,(G^R)^{zz}_{xy}]_x + [(P\phi)_x\,(G^R)^{zz}_{xy}]_y - [(1-P)\,\phi\, G^R_x]_x$$

The potential representation (17) can then be expressed as

$$\begin{aligned}\widetilde{\phi}^R = &\int_{\Sigma_B} d\mathcal{A}\,[\,G^R\,\nabla\phi - \mathbf{G^R}\times\nabla(P\phi) - (1-P)\,\phi\,\nabla G^R\,]\cdot\mathbf{n}\\
&+\int_\Gamma d\mathcal{L}\,\{\,F^2\phi_x\,t^y G^R + f^2 P\phi\,[\,t^y(G^R)^{zz}_x - t^x(G^R)^{zz}_y\,]\\
&\qquad - (P\phi)_t\,(F^2 G^R_x - i\widehat{\tau}G^R)^{zz}_y - (1-P)\,\phi\, t^y\,(F^2 G^R_x - i\widehat{\tau}G^R)\,\}\\
&+\int_{\Sigma_0} dx\,dy\,[\,(P\phi)_x\,(\pi^R)^{zz}_x + (P\phi)_y\,(\pi^R)^{zz}_y + (1-P)\,\phi\,\pi^R - G^R\,\pi^\phi\,]\end{aligned} \tag{19}$$

where Stokes' theorem and the relation $\nabla(P\phi)\cdot\mathbf{t} = (P\phi)_t$ were used. The integrands in the potential representation (19) with (13) are asymptotically equivalent to those related to the classical ($P = 0$) and weakly-singular ($P = 1$) representations in the farfield $r \to \infty$ and the nearfield $r \to 0$, respectively.

The Fourier component $\widetilde{\phi}^F$ and the wave component $\widetilde{\phi}^W$ in (16) are similarly given by (19) and (18), in which the superscript R only needs to be replaced by F and W, respectively. The Fourier-Kochin approach can be used to express the Fourier component $\widetilde{\phi}^F$ in terms of the double Fourier integral

$$\widetilde{\phi}^F = \lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{-\infty}^{\infty}d\beta\int_{-\infty}^{\infty}d\alpha\; SA\,\frac{e^{\widetilde{z}\,k - i(\widetilde{x}\,\alpha + \widetilde{y}\,\beta)}}{D + i\,\varepsilon\, D_1} \quad\text{with}\quad \left\{\begin{aligned} A &= e^{-F^2 k}(1 - e^{-k/f^2})\,D/k - 1\\ D &= k - (F\alpha - f)^2\\ D_1 &= F\alpha - f\end{aligned}\right\} \tag{20a}$$

The spectrum function $S(\alpha\,,\beta)$ in (20a) is given by distributions of elementary waves $e^{\,k\,z + i\,(\alpha\,x + \beta\,y)}$:

$$S = \int_{\Sigma_B} d\mathcal{A}(\nabla\phi\cdot\mathbf{n} - i\,A^S_B)\,e^{k\,z}E + \int_\Gamma d\mathcal{L}(F^2\phi_x\,t^y + i\,A^S_\Gamma)\,E + \int_{\Sigma_0} dx\,dy\,(A^S_0\,\frac{D}{k} - \pi^\phi)\,E \tag{20b}$$

with $E = e^{\,i\,(\alpha\,x + \beta\,y)}$. Similarly, the wave component $\widetilde{\phi}^W$ in (16) is given by *single* Fourier integrals along the dispersion curves $D = 0$, as in [2]. The amplitude functions in (20b) are defined as

$$A^S_B = \frac{\alpha(P\phi)_x + \beta\,(P\phi)_y}{k}\,n^z - \frac{\alpha\,n^x + \beta\,n^y}{k}(P\phi)_z + k\,(1-P)\,\phi\left(\frac{\alpha\,n^x + \beta\,n^y}{k} - i\,n^z\right) \tag{21a}$$

$$A^S_\Gamma = f^2\,\frac{P\phi}{k}\,\frac{\alpha\,t^y - \beta\,t^x}{k} - \frac{F^2\alpha - \widehat{\tau}}{k}\left(k\,(1-P)\,\phi\,t^y + i\,\frac{\beta}{k}(P\phi)_t\right) \tag{21b}$$

$$A^S_0 = k\,(1-P)\,\phi + i\,\frac{\alpha\,(P\phi)_x + \beta\,(P\phi)_y}{k} \tag{21c}$$

In the limits $k \to 0$ (left half) and $k \to \infty$ (right half), the amplitude functions (21) behave as

$$\left\{\begin{matrix} A^S_B = O(k) & A^S_\Gamma = O(1) & A^S_0 = O(k)\\ A^S_B = O(1) & A^S_\Gamma = O(1/k) & A^S_0 = O(1)\end{matrix}\right\} \qquad \left\{\begin{matrix} A^S_B = O(k) & A^S_\Gamma = O(k) & A^S_0 = O(k)\\ A^S_B = O(1) & A^S_\Gamma = O(1) & A^S_0 = O(1)\end{matrix}\right\}$$

if $P = 0$ (top row) and $P = 1$ (bottom row). Thus, the spectrum functions associated with the classical ($P = 0$) and weakly-singular ($P = 1$) potential representations are preferable in the limits $k \to 0$ and $k \to \infty$, respectively. The weight function

$$P = k^2/(k^2 + k_*^2) \qquad \text{where } k_* \text{ corresponds to a transition wavenumber}$$

yields functions $A^S_B\,, A^S_\Gamma\,, A^S_0$ in the Fourier-Kochin representation (20) of the Fourier component $\widetilde{\phi}^F$ in the decomposition (16) that are asymptotically equivalent to the spectrum functions related to the corresponding classical and weakly-singular representations as $k \to 0$ and $k \to \infty$, respectively.

[1] Noblesse F., Yang C. (2004) *Weakly-singular boundary-integral representations of free-surface flows about ships or offshore structures*, Jl Ship Research **48**

[2] Noblesse F., Yang C. (2004) *A simple Green function for diffraction-radiation of time-harmonic waves with forward speed*, Ship Technology Research **51**:35-52