Flexural-gravity wave diffraction by finite cracks of arbitrary shape in ice sheets

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Introduction

In this paper we consider the diffraction of plane incident waves from infinity by a configuration of arbitrarily-shaped narrow cracks in a thin elastic plate which covers the entire surface of a fluid of constant depth. This problem has application to cracks in ice sheets and also to cracks that may occur in very large floating platforms, such as those being considered for use as an offshore runway in Japan. The present work builds on that of Porter & Evans (2004) who considered oblique wave scattering by an arbitrary number, N, of infinitely-long parallel straight line cracks in elastic plates on water. That problem, which is essentially two-dimensional, was solved by deriving certain pairs of source functions for a single crack and using linear superposition to formulate the solution of the N-crack problem in terms of a $2N \times 2N$ system of linear algebraic equations involving coefficients P_i and Q_i, $i = 1, \ldots, N$ which relate to the jumps in displacement and gradient across each of the N cracks.

The particular approach described above is much more elegant than other approaches to problems involving infinite parallel cracks such as those presented by Evans & Porter (2003) and Squire & Dixon (2001) who used non-orthogonal eigenfunction expansions and Green's functions. Furthermore, the source function approach has the advantage that it can be extended to more general problems where other methods cannot be applied.

Thus, in the much more complicated problem being considered here, we find that similar ideas can be developed. That is, a pair of source functions can be derived for the fully three-dimensional problem involving N arbitrarily-shaped cracks of finite length in the elastic plate. It is shown how the solution can to be formulated as a $2N \times 2N$ coupled system of integral equations involving unknown functions $P_i(s)$ and $Q_i(s)$, $i = 1, \ldots, N$ which again represent the jumps in plate elevation and gradient across each of the cracks as a function of s, the arc length along each crack. These functions are determined by imposing a pair of conditions on the free edges of each of the cracks.

For simplicity, we only derive the theory for a single arbtirary crack ($N = 1$) in an elastic plate; once this is established, the extension larger numbers of cracks is straightforward.

Formulation of the problem

Cartesian coordinates x, y, z are chosen with z directed vertically upwards and $z = 0$ coinciding with the lower surface of the undisturbed elastic plate which has a small constant thickness d. The fluid occupies the region $-h < z < 0$, $(x, y) \in D_h$ where $D_h = \{-\infty < x, y < \infty\}$. There is a single crack in the elastic plate denoted by the curve C which is described parametrically by $C = \{\boldsymbol{\rho}(u),\ -1 < u < 1\}$ where $\boldsymbol{\rho}(u) = (x(u), y(u))$.

In the fluid, the usual assumptions of linearised theory apply so that there exists a velocity potential $\Phi(\boldsymbol{r}, t)$ where $\boldsymbol{r} = (x, y, z)$ and the time dependence t is removed by assuming a harmonic dependence of angular frequency ω such that $\Phi(\boldsymbol{r}, t) = \Re\{-\mathrm{i}\omega\phi(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t}\}$. Thus, for $(x, y) \in D_h$,

$$\nabla^2\phi = 0, \quad -h < z < 0, \qquad \text{and} \qquad \phi_z = 0, \quad z = -h. \tag{1}$$

The elevation of the upper surface of the fluid, and hence the deflection of the plate, is defined by $\Re\{\eta(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}\omega t}\}$ where $\boldsymbol{\rho} = (x, y)$ and

$$\eta(\boldsymbol{\rho}) = \phi_z|_{z=0}, \quad \boldsymbol{\rho} \in D_h \backslash C. \tag{2}$$

This equation is supplemented by an equation describing the motion of the plate, which is derived on the basis of thin plate theory and given by

$$(\mathcal{L}\phi)(\boldsymbol{\rho}) \equiv (D\nabla_h^4 + 1 - \delta)\eta - \kappa\phi|_{z=0} = 0, \quad \boldsymbol{\rho} \in D_h \backslash C \tag{3}$$

in which $\nabla_h = (\partial/\partial x, \partial/\partial y, 0)$, whilst $D = Ed^3/(12\rho_w g(1 - \nu^2))$ where E is Young's Modulus, ν is Poisson's ratio, ρ_w is the density of water and g is gravitational accleration and $\delta = (\rho_p/\rho_w)\kappa d$ where $\kappa = \omega^2/g$ and ρ_p is the density of the plate.

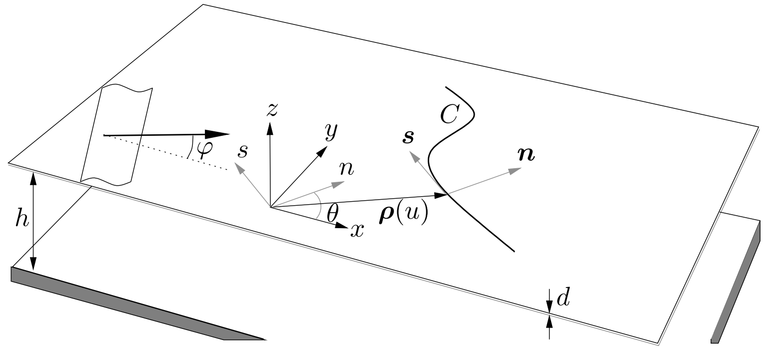

Figure 1: Coordinate system describing the crack C in a plate on water of depth h

Along the curve C, let $\boldsymbol{n}$ and $\boldsymbol{s}$ describe the normal and tangential unit vectors to the curve where $\boldsymbol{n}$ has direction cosines $(\cos\theta, \sin\theta, 0)$ with respect to the fixed cartesian frame where $\theta = \theta(s)$ and s measures the arc length along the curve (see figure 1). Then $\boldsymbol{n} = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta$ and $\boldsymbol{s} = -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta$. It follows that

$$\nabla_h^2 = \frac{\partial^2}{\partial n^2} + \frac{\partial^2}{\partial s^2} + \theta'(s)\frac{\partial}{\partial n}, \qquad \text{and} \qquad \frac{\partial}{\partial n}\frac{\partial}{\partial s} - \frac{\partial}{\partial s}\frac{\partial}{\partial n} + \theta'(s)\frac{\partial}{\partial s} = 0 \tag{4}$$

(see de Veubeke (1979)) where $\partial/\partial n = \boldsymbol{n}.\nabla_h$, $\partial/\partial s = \boldsymbol{s}.\nabla_h$ and $\theta'(s)$ is the curvature of C.

In this local coordinate system based on the curve C, the boundary conditions of zero bending moment and shearing stresses are given respectively (de Veubeke (1979)) by

$$(\mathcal{B}\eta)(\boldsymbol{\rho}) \equiv \left(\nabla_h^2 - \nu_1\left(\frac{\partial^2}{\partial s^2} + \theta'(s)\frac{\partial}{\partial n}\right)\right)\eta \to 0, \quad \text{on } s = s(u) \text{ as } n \to n(u)^{\pm} \tag{5}$$

and

$$(\mathcal{S}\eta)(\boldsymbol{\rho}) \equiv \left(\frac{\partial}{\partial n}\nabla_h^2 + \nu_1\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \theta'(s)\frac{\partial}{\partial s}\right)\right)\eta \to 0, \qquad \text{on } s = s(u) \text{ as } n \to n(u)^{\pm} \tag{6}$$

for $-1 < u < 1$ where $\nu_1 = 1 - \nu$ and where (n, s) is the coordinate system based on the cartesian origin, aligned with the vectors $\boldsymbol{n}$ and $\boldsymbol{s}$. The curve C is described parametrically in terms of this system as $\boldsymbol{\rho}(u) = (n(u), s(u))$, $-1 < u < 1$.

As a consequence of (5) and (6) we have the jump conditions

$$[(\mathcal{B}\eta)(\boldsymbol{\rho})] = [(\mathcal{S}\eta)(\boldsymbol{\rho})] = 0, \qquad \text{where} \qquad [u(\boldsymbol{\rho})] = \lim_{n\to n(u)^+}\{u(\boldsymbol{\rho})|_{s=s(u)}\} - \lim_{n\to n(u)^-}\{u(\boldsymbol{\rho})|_{s=s(u)}\}. \tag{7}$$

In addition, at the end points of the curve C no concentrated corner forces exist, from which the condition

$$\left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \theta'(s)\frac{\partial}{\partial s}\right)\eta = 0 \tag{8}$$

applies and, by ensuring boundedness of the strain energy in the elastic plate at the ends of the crack, it can be shown that $[\eta] \sim \rho^{3/2}$, $[\partial\eta/\partial n] \sim \rho^{1/2}$ and $[\partial\eta/\partial s] \sim \rho^{1/2}$ as $\rho \to 0$ where ρ measures the distance in the plane $z = 0$ from either end point of C.

The incident wave potential for a plane wave from infinity propagating at an angle φ with respect to the positive x-direction is given by

$$\phi_0(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}k_0(x\cos\varphi + y\sin\varphi)}Y_0(z) \tag{9}$$

where k_0 is the positive real root of the dispersion relation

$$K(k_n) \equiv (Dk_n^4 + 1 - \delta)k_n \sinh k_n h - \kappa \cosh k_n h = 0 \tag{10}$$

with $n = 0$ and $Y_n(z) = \cosh k_n(z + h)$ are non-orthogonal depth eigenfunctions. The other roots of (10) include four complex roots, which are denoted by $k_{-1} = p + \mathrm{i}q$, $k_{-2} = -p + \mathrm{i}q$ where $p, q > 0$ and their complex conjugates and an infinite sequence of imaginary roots, $\pm k_n$, $n = 1, 2, \ldots$ defined such that $\Im\{k_n\} > 0$.

The functions $\phi - \phi_0$ and $\eta - \eta_0$ where $\eta_0 = \partial\phi_0/\partial z|_{z=0}$ both satisfy the Sommerfeld radiation condition.

Source functions and solution

The key to solving the scattering problem defined in the previous section is in the construction of a pair of 'source functions' which act along the curve C. Thus we define functions $\psi_i(\boldsymbol{r}; \boldsymbol{\rho}(v))$, $i = 1, 2$ satisfying

$$\nabla^2 \psi_i = 0, \quad -h < z < 0, \qquad \text{and} \qquad \partial\psi_i/\partial z = 0, \quad z = -h \tag{11}$$

with

$$(\mathcal{L}\psi_i)(\boldsymbol{\rho}; \boldsymbol{\rho}(v)) \equiv (D\nabla_h^4 + 1 - \delta)w_i - \kappa\psi_i|_{z=0} = 0, \quad \text{for } \boldsymbol{\rho} \neq \boldsymbol{\rho}(v) \tag{12}$$

where

$$w_i(\boldsymbol{\rho}; \boldsymbol{\rho}(v)) = \partial\psi_i/\partial z|_{z=0}. \tag{13}$$

We also require that ψ_i and hence w_i represent outgoing waves as $|\boldsymbol{\rho}| \to \infty$. Finally we impose the following jump conditions as defined by (7)

$$\left.\begin{aligned} &[w_1] = 0 \\ &[\partial w_1/\partial n] = \delta(s(u) - s(v)) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} &[w_2] = \delta(s(u) - s(v)) \\ &[\partial w_2/\partial n] = 0 \end{aligned}\right\} \tag{14}$$

in addition to

$$[(\mathcal{B}w_i)(\boldsymbol{\rho}; \boldsymbol{\rho}(v))] = [(\mathcal{S}w_i)(\boldsymbol{\rho}; \boldsymbol{\rho}(v))] = 0, \quad -1 < u, v < 1 \text{ for } i = 1, 2. \tag{15}$$

With the definition of the functions ψ_i, $i = 1, 2$, it is straightforward to confirm that the general solution of the scattering problem satifying (1), (3) and (7) in addition to the radiation condition, is

$$\phi(\boldsymbol{r}) = \phi_0(\boldsymbol{r}) + \int_{-1}^{1} \Big\{P(v)\psi_1(\boldsymbol{r}; \boldsymbol{\rho}(v)) + Q(v)\psi_2(\boldsymbol{r}; \boldsymbol{\rho}(v))\Big\}|s'(v)|\mathrm{d}v \tag{16}$$

from which it follows that

$$\eta(\boldsymbol{\rho}) = \eta_0(\boldsymbol{\rho}) + \int_{-1}^{1} \Big\{P(v)w_1(\boldsymbol{\rho}; \boldsymbol{\rho}(v)) + Q(v)w_2(\boldsymbol{\rho}; \boldsymbol{\rho}(v))\Big\}|s'(v)|\mathrm{d}v \tag{17}$$

where the definitions

$$P(u) = [\partial\eta/\partial n], \qquad \text{and} \qquad Q(u) = [\eta] \tag{18}$$

follow from applying (14) to (17) and represent the jumps in the gradients and displacements across the curve C as a function of the length of the crack. The functions $P(u)$ and $Q(u)$ are yet to be determined, and this is done by applying the two edge conditions (5) and (6) to (17) which gives

$$\left.\begin{aligned} -(\mathcal{B}\eta_0)(\boldsymbol{\rho}(u)) &= \int_{-1}^{1} \Big\{P(v)(\mathcal{B}w_1)(\boldsymbol{\rho}(u); \boldsymbol{\rho}(v)) + Q(v)(\mathcal{B}w_2)(\boldsymbol{\rho}(u); \boldsymbol{\rho}(v))\Big\}|s'(v)|\mathrm{d}v, \\ -(\mathcal{S}\eta_0)(\boldsymbol{\rho}(u)) &= \int_{-1}^{1} \Big\{P(v)(\mathcal{S}w_1)(\boldsymbol{\rho}(u); \boldsymbol{\rho}(v)) + Q(v)(\mathcal{S}w_2)(\boldsymbol{\rho}(u); \boldsymbol{\rho}(v))\Big\}|s'(v)|\mathrm{d}v. \end{aligned}\right\} \tag{19}$$

It remains to determine the functions $\psi_i(\boldsymbol{r}; \boldsymbol{\rho}(v))$, $i = 1, 2$ from the definitions (11)–(15). This is done using Fourier transforms in x and y, a complicated process which involves the use of Green's identity applied to the biharmonic operator to reduce the double integral to one along the curve C involving an integrand representing jumps of certain functions. Use of the boundary operators $\mathcal{B}$ and $\mathcal{S}$ defined by (5) and (6) and repeated integration by parts which expose free terms which are eliminated using (8) and the boundedness conditions on η and its derivatives at the end points of C described thereafter. This process belies the simplicity of the outcome, namely in the definitions

$$\psi_1(\boldsymbol{r}; \boldsymbol{\rho}(v)) = -D(\mathcal{B}\chi), \qquad \text{and} \qquad \psi_2(\boldsymbol{r}; \boldsymbol{\rho}(v)) = D(\mathcal{S}\chi) \tag{20}$$

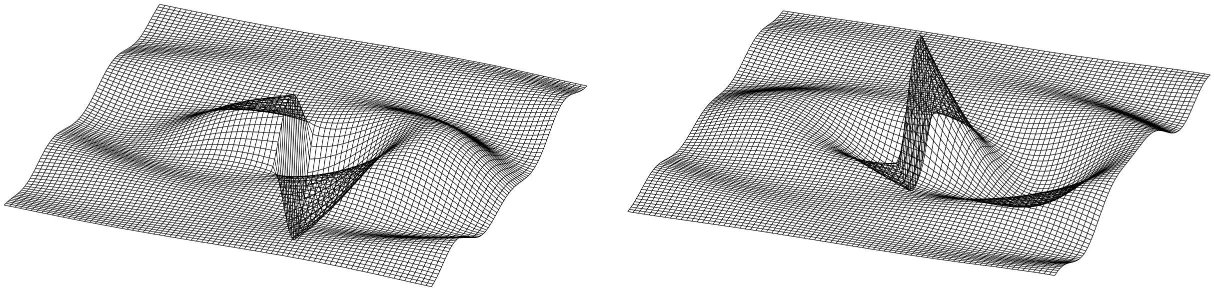

Figure 2: Two snaphots in time for the diffracted wave field for a crack with $k_0 a = 4$.

where

$$\chi(\boldsymbol{r};\boldsymbol{\rho}(v)) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cosh k(z+h)}{K(k)}\mathrm{e}^{\mathrm{i}\alpha(x(v)-x)}\mathrm{e}^{\mathrm{i}\beta(y(v)-y)}\mathrm{d}\alpha\mathrm{d}\beta \tag{21}$$

where $k^2 = \alpha^2 + \beta^2$. The function χ can also be evaluated as a series by deforming the contour of integration into the appropriate part of the plane and picking up the residues from the zeros of $K(k)$ given by (10). Thus it can be shown that

$$\chi(\boldsymbol{r};\boldsymbol{\rho}(v)) = \frac{\mathrm{i}}{4}\sum_{n=-2}^{\infty}\frac{Y_n(z)Y_n'(0)}{C_n}H_0(k_n R) \tag{22}$$

where $R = \sqrt{(x(v)-x)^2 + (y(v)-y)^2}$, $C_n = \frac{1}{2}(\kappa h + (5D^4 + 1 - \delta)\sinh^2 k_n h)$ and H_0 is the Hankel function of the first kind. It may be noticed that $\chi(\boldsymbol{r};\boldsymbol{\rho}(v))$ is just the Green's function for a point source of forcing on the plate $z = 0$ at $\boldsymbol{\rho} = \boldsymbol{\rho}(v)$.

Numerical procedure and results

The coupled system of integral equations (19) are to be solved for functions $P(v)$ and $Q(v)$, $-1 < v < 1$. This is done by expanding them in the appropriate weighted set of functions which reflects the local behaviour at the end points of the curve C and applying a Galerkin procedure to transform the integral equations to a linear system of equations. For $P(v)$, which represents $[\partial\eta/\partial n]$, the weighting that reflects the local behaviour at the end points is $(1-v^2)^{1/2}$ and the corresponding set of functions are the Chebychev polynomials of the second kind, $U_n(v)$, $n = 0, 1, \ldots$. In the case of $Q(v)$, which represents $[\eta]$, a weighting of $(1-v^2)^{3/2}$ is anticipated and the corresponding set of functions are $C_n^{(2)}(v)$, $n = 0, 1, \ldots$ being Gegenbauer polynomials.

Despite the elegance of the solution, the kernels of the integrand in (19), $(\mathcal{B}w_i)$ and $(\mathcal{S}w_i)$, $i = 1, 2$ are not straightforward to compute for general curves because of the complexity of the operators $\mathcal{B}$ and $\mathcal{S}$. In figure 2 we have considered a straight finite length crack. In this case $\mathcal{B}$ and $\mathcal{S}$ simplify and further analytic progress is possible (such as identifying symmetric and antisymmetric parts of the solution) which simplifies the integral equations. In figure 2, a wave is normally-incident on a straight crack of length $2a$ with incident wavenumber $k_0 a = 4$ (alternatively, the wavelength of the incident wave is approximately three-quarters the length of the crack). The diffracted wave field is shown for two different times in a period of motion. Although no vertical scale is shown, significantly the maximum amplitude of the crack displacement is four times the height of the incident wave.

A selection of other numerical results will be shown at the workshop.

References

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