Scattering of obliquely incident waves by submerged ridges

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Introduction

In this paper we apply the full linear theory to the scattering of obliquely incident monochromatic waves by a submerged infinite ridge of uniform cross section as a specific example of a more general theory developed by the authors. Porter & Porter [2000] solved the scattering problem for normally incident waves by an arbitrary topography retaining an exact formulation throughout and solving the resultant integral equation by an extremely accurate Galerkin method. Crucial to this technique was the conversion of normal derivatives to tangential derivatives, in essence by an application of Cauchy-Riemann style equations. Unfortunately this restricted the approach to strictly two-dimensional geometries with no obvious means of extending to quasi 2D or fully 3D problems. The authors have developed fully 3D analogue of this method which is based on exchanging the scalar Green's function for a vector Green's function using a transformation suggested in Noblesse [2004] that allows the full linear theory to be applied to arbitrary topographies. In the case of normal incidence we recover exactly the formulation from Porter & Porter [2000]. At the workshop we will also present results for scattering by axisymmetric seamounts for which it is worth noting that Chamberlain & Porter [JFM 338], in solving this latter problem by modified mild-slope equations, stated that *an investigation of this problem using the full linear theory would be formidable!*

Formulation and preliminaries

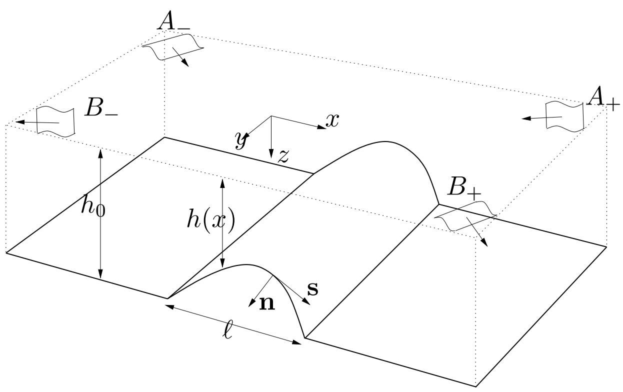

Figure 1: Geometrical description of the problem

Cartesian coordinates (x, y, z) are chosen with the x and y axes lying in the undisturbed free surface of the fluid and z directed vertically downwards. The topography consists of an infinitely-long ridge with constant cross-section in the (x, z)-plane and which protrudes from an otherwise flat bed of depth h_0. The y axis is chosen to be parallel to the generating axis of the ridge and the fluid is bounded below by the curve $\Gamma : \{z = h(x), -\infty < x, y < \infty\}$ which generates the ridge and where $h(x)$ is assumed to be a continuous function with $h(x) = h_0$, a constant, for $x \notin (0, \ell)$. Furthermore, it is assumed that $h(x) \le h_0$ for $x \in (0, \ell)$. On an arbitrary section of the lower boundary of the fluid Γ, we define normal and tangential vectors by

$$\left.\begin{aligned} \mathbf{n} &= (-h'(x), 0, 1)/\sigma(x) \\ \mathbf{s} &= (1, 0, h'(x))/\sigma(x) \end{aligned}\right\}, \qquad \sigma(x) = \sqrt{1 + [h'(x)]^2} \tag{1}$$

respectively.

We assume linearised water wave theory, in which the fluid is inviscid and incompressible and the motion is irrotational and of small amplitude, so that the flow can be described in terms of a velocity potential . We also assume that the incident wave is time-harmonic and has exponential dependence in y namely e^{ily}, therefore as the geometry does not vary in y this exponential variation is inherited by the potential. Therefore the potential is given by $\Re\{\phi(x, z)e^{ily}e^{-i\omega t}\}$ and ϕ satisfies

$$(\nabla^2 - l^2)\phi = 0 \text{ for } \{x, z\} \in D, \qquad \mathbf{n}.\nabla\phi = 0 \text{ for } \{x, z\} \in \Gamma \text{ and } \frac{\partial \phi}{\partial z} + K\phi = 0, \text{ on } z = 0\,, \tag{2}$$

where $D : \{0 < z < h(x), -\infty < x < \infty\}$ is the fluid domain, $K = \omega^2/g$ and g is gravitational acceleration. To complete the formulation of the problem, we need radiation conditions at infinity, which are written as

$$\phi(x, z) \sim \begin{cases} A_-\phi_0^+(x, z) + B_-\phi_0^-(x, z), & x \to -\infty, \\ A_+\phi_0^-(x, z) + B_+\phi_0^+(x, z), & x \to \infty. \end{cases} \tag{3}$$

Here $\phi_0^{\pm}(x,z)$ define waves propagating obliquely towards $x=\pm\infty$ in water of constant depth h_0, whilst $A_\pm$ and $B_\pm$ represent wave amplitudes associated with waves that are incoming and outgoing (respectively) on the ridge from $x=\pm\infty$. More specifically,

$$\phi_0^{\pm}(x,z) = \mathrm{e}^{\pm\mathrm{i}\alpha x}\psi_0(z) \tag{4}$$

where $\alpha = k\sin\theta$ and $l = k\cos\theta$ are components of the wavenumber, k, in the x and y directions (respectively) for a wave propagating at an angle $\pm\theta$ with respect to the positive y-axis. Also we define depth modes for $r=0,1,\ldots,$ by

$$\psi_r(z) = N_r^{-1/2}\cos k_r(h_0-z), \qquad N_r = \frac{1}{2}\left(1+\frac{\sin 2k_r h_0}{2k_r h_0}\right), \tag{5}$$

with $k_0 = -\mathrm{i}k$ where k is the positive root of the dispersion relation $K = k\tanh kh_0$ defining the wavenumber k in terms of frequency and depth, and $\{k_r\}$ are its complex roots.

The reflection and transmission coefficients due a wave of unit amplitude incident from $\pm\infty$, denoted by $R_\pm$ and $T_\pm$, respectively, are defined in terms of $A_\pm$ and $B_\pm$ with

$$\begin{pmatrix} B_+ \\ B_- \end{pmatrix} = \mathsf{S}\begin{pmatrix} A_- \\ A_+ \end{pmatrix}, \qquad \text{where} \qquad \mathsf{S} = \begin{pmatrix} T_- & R_+ \\ R_- & T_+ \end{pmatrix} \tag{6}$$

The matrix S is usually referred to as the scattering matrix and is regarded as the principal unknown in this problem.

The method of solution relies on the use of a Green's function appropriate to this problem. Thus we define $G(x,z;x_0,z_0)$ where (x_0,z_0) is regarded as the field point and (x,z) the source point, satisfying

$$(\nabla^2 - l^2)G(x,z;x_0,z_0) = -\delta(x-x_0)\delta(z-z_0) \quad \text{in } 0<z,z_0<h_0, \tag{7}$$

$$\frac{\partial G}{\partial z} + KG = 0 \quad \text{on } z=0 \qquad \text{and} \qquad \frac{\partial G}{\partial z} = 0 \quad \text{on } z=h_0 \tag{8}$$

holding for $-\infty < x, x_0 < \infty$. Then it may be shown that

$$G(x,z;x_0,z_0) = \sum_{r=0}^{\infty} \frac{\psi_r(z)\psi_r(z_0)}{2\alpha_r h_0}\mathrm{e}^{-\alpha_r|x-x_0|} \tag{9}$$

where for $r=0,1,\ldots,$ $\alpha_r = \sqrt{k_r^2+l^2}$ and $\alpha_0 = -\mathrm{i}\sqrt{k^2-l^2}$ which gives $\alpha_0 = -\mathrm{i}\alpha$.

We will find it convenient to write G in the form

$$G = G_0 + \widehat{G} \tag{10}$$

where

$$G_0(x,z;x_0,z_0) = \frac{\mathrm{i}\psi_0(z)\psi_0(z_0)}{2\alpha h_0}\cos\alpha(x-x_0) \tag{11}$$

is the separable component of the wave-like part of the Green's function whilst

$$\widehat{G}(x,z;x_0,z_0) = -\frac{\psi_0(z)\psi_0(z_0)}{2\alpha h_0}\sin\alpha|x-x_0| + \sum_{r=1}^{\infty}\frac{\psi_r(z)\psi_r(z_0)}{2\alpha_r h_0}\mathrm{e}^{-\alpha_r|x-x_0|}. \tag{12}$$

is the remainder of G.

At this point we define functions related to the depth eigenfunctions $\psi_r(z)$ which will play an important role in our formulation. Thus we define

$$\chi_r(z) = -k_r\int_{h_0}^{z}\psi_r(z')dz' = N_r^{-1/2}\sin k_r(h_0-z), \qquad r=0,1,2,\ldots \tag{13}$$

which are precisely the functions defined in a purely two-dimensional wave scattering problem considered by Porter & Porter [2000], although their introduction was motivated by certain relations based on the Cauchy-Riemann equations.

Derivation of an integral equation

In this section, we set out to develop an exact formulation in terms of integral equations of the solution to the problem, as a means of calculating the scattering matrix S. The first step is to apply Green's Identity to the functions $\phi(x,z)$ and $G(x,z;x_0,z_0)$ to obtain

$$\phi(x_0,z_0) = A_-\phi_0^+(x_0,z_0) + A_+\phi_0^-(x_0,z_0) - \int_\Gamma \phi(x,z)\frac{\partial}{\partial n}G(x,z;x_0,z_0)\,\mathrm{d}s. \tag{14}$$

In (14) terms $A_\pm$ are as expected representing incoming waves from infinity, and the integral is restricted to Γ due to the construction of G. At this point, if we moved the field point onto the boundary $(x, h(x))$, (14) becomes a second kind integral equation for ϕ for points on Γ. Instead of pursuing this course of action any further, we develop the formulation in an analogous manner to that appearing in the two-dimensional scattering problem considered by Porter & Porter (2000), anticipating a self-adjoint structure in the final integral equations that is not enjoyed by an integral equation arising directly from (14). We define normal and tangential vectors, $\{\mathbf{n}_0, \mathbf{s}_0\}$ analagously to (1) extending their definition into D. We then apply the operator $\mathbf{n}_0.\nabla_0$ to (14) to obtain

$$\frac{\partial}{\partial n_0}\phi(x_0, z_0) = A_- \frac{\partial}{\partial n_0}\phi_0^+(x_0, z_0) + A_+ \frac{\partial}{\partial n_0}\phi_0^-(x_0, z_0) - \frac{\partial}{\partial n_0}\int_\Gamma \phi(x, z)\frac{\partial}{\partial n}G(x, z; x_0, z_0)\ \mathrm{d}s. \tag{15}$$

It turns out that the following relations can be established

$$\frac{\partial^2}{\partial n_0 \partial n}G = \frac{\partial^2}{\partial s_0 \partial s}G^{zz_0}_{xx_0} - \frac{l^2}{\sigma}\frac{\partial}{\partial s_0}G^{zz_0}_{x_0} - \frac{l^2}{\sigma_0}\frac{\partial}{\partial s}G^{zz_0}_{x} + \frac{l^4}{\sigma\sigma_0}G^{zz_0} \tag{16}$$

and

$$\frac{\partial}{\partial n_0}\phi_0^\pm = F^\pm(s_0) \tag{17}$$

where

$$F_\pm(x, z) = \left(\mp\frac{\alpha}{k}\frac{\partial}{\partial s} - \frac{\mathrm{i}l^2}{k\sigma(x)}\right) f_\pm(x, z) \quad \text{and} \quad f_\pm(x, z) = \mathrm{e}^{\pm\mathrm{i}\alpha x}\chi_0(z) \tag{18}$$

and we have used Noblesse's notation

$$G^z_x \equiv \int_{h_0}^{z} \frac{\partial}{\partial y}G(x, z'; x_0, z_0)\,\mathrm{d}z' \quad \text{etc} \ \ldots \tag{19}$$

This step is crucial and, although by no means obvious, it arises naturally as a result of the general theory - we will discuss this further at the workshop. Therefore, using (16) and (17) we are able to rewrite (15) as

$$\begin{aligned}\frac{\partial}{\partial n_0}\phi(x_0, z_0) = A_-\ F^+ \ +\ A_+\ F^- - \frac{\partial}{\partial s_0}\int_\Gamma \left(\frac{\partial}{\partial s}G^{zz_0}_{xx_0} - \frac{l^2}{\sigma}G^{zz_0}_{x_0}\right)\phi(s)\,\mathrm{d}s \\ + \frac{l^2}{\sigma_0}\int_\Gamma \left(\frac{\partial}{\partial s}G^{zz_0}_{x} - \frac{l^2}{\sigma}G^{zz_0}\right)\phi(s)\,\mathrm{d}s, \quad \text{where } s = (x, z) \in \Gamma.\end{aligned} \tag{20}$$

At this point some rearrangement of this last expression is needed to isolate terms which are separable (i.e. associated with G_0) as the ultimate goal is to obtain a self-adjoint integral operator equation. When this rearrangement is carried out and moving (x_0, z_0) onto Γ so that the left had side of (20) vanishes it turns out that

$$\begin{aligned}0 = \tfrac{1}{2}(A_- + B_+)F_+(s_0) + \tfrac{1}{2}(A_+ + B_-)F_-(s_0) \\ - \frac{\partial}{\partial s_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}^{zz_0}_{xx_0} - \frac{l^2}{\sigma}\widehat{G}^{zz_0}_{x_0}\right)\phi(s)\,\mathrm{d}s + \frac{l^2}{\sigma_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}^{zz_0}_{x} - \frac{l^2}{\sigma}\widehat{G}^{zz_0}\right)\phi(s)\,\mathrm{d}s.\end{aligned} \tag{21}$$

Thus (21) now represents an integro-differential equation for the function ϕ. So if we define the integro-differential operator in (21) as

$$(\mathcal{K}\phi)(s) \equiv \frac{\partial}{\partial s_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}^{zz_0}_{xx_0} - \frac{l^2}{\sigma}\widehat{G}^{zz_0}_{x_0}\right)\phi(s)\,\mathrm{d}s - \frac{l^2}{\sigma_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}^{zz_0}_{x} - \frac{l^2}{\sigma}\widehat{G}^{zz_0}\right)\phi(s)\,\mathrm{d}s. \tag{22}$$

and a pair of functions $\phi^\pm(s)$ is defined on Γ to satisfy

$$(\mathcal{K}\phi^\pm)(s_0) = F_\pm(s_0), \qquad s_0 \in \Gamma, \tag{23}$$

then it follows that the solution of (21) is given by

$$\phi(s) = \tfrac{1}{2}(A_- + B_+)\phi^+(s) + \tfrac{1}{2}(A_+ + B_-)\phi^+(s). \tag{24}$$

We now introduce the inner product notation for functions $u(s), v(s) \in \mathcal{H}$ (where $\mathcal{H}$ is the space of functions whose derivatives belong to $L_2(\Gamma)$)

$$\langle u, v\rangle = \int_\Gamma u\overline{v}\ \mathrm{d}s \tag{25}$$

and then define

$$P_{\pm}^{\pm} = \langle \Phi^{\pm}, F_{\pm} \rangle, \qquad \text{and} \qquad \lambda = \frac{\mathrm{i}}{4\alpha h_0} \tag{26}$$

where the superscripts and subscripts on the left-hand side correspond to those that are attached to quantities on the right-hand side. Using (24) it transpires that the far field amplitudes are related by the expression

$$(\mathsf{I} + \lambda \mathsf{P}) \begin{pmatrix} B_+ \\ B_- \end{pmatrix} = (\mathsf{I} - \lambda \mathsf{P}) \begin{pmatrix} A_- \\ A_+ \end{pmatrix}, \qquad \text{where} \qquad \mathsf{P} = \begin{pmatrix} P_+^+ & P_+^- \\ P_-^+ & P_-^- \end{pmatrix} \tag{27}$$

where I is the 2×2 identity matrix, an upon comparison with (9) we deduce that $\mathsf{S} = (\mathsf{I} + \lambda \mathsf{P})^{-1} (\mathsf{I} - \lambda \mathsf{P})$.

Solution and results

Although our formulation has remained exact so far, we must inevitably solve it numerically. We therefore solve the integral equation by establishing a variational principle equivalent to the Rayleigh-Ritz method. The solutions of the integral equation $\phi^{\pm}$ are approximated by

$$\phi^{\pm}(s) \simeq \widehat{\phi}^{\pm}(s) = \sum_{n=1}^{N} a_n^{\pm} p_n(s), \tag{28}$$

where the trial functions $p_n(s)$ are chosen to model the local fluid behaviour at the end points of the topography. In this case the variational principle yields the condition

$$\sum_{n=1}^{N} a_n^{\pm} \langle \mathcal{K} p_n, p_m \rangle = \langle F_{\pm}, p_m \rangle \tag{29}$$

which is a $N \times N$ linear system of equations for $a_n^{\pm}$. The structure of the problem is such that all y dependence is removed and also by assuming $h(x)$ is a single-valued function we may project all integrals onto the x axis. We are also able to transfer the tangential derivatives to the trial functions by integrating by parts and noting that $\hat{g}^{zz_0}$ vanishes at the end points where $z = h_0$ The system matrix has four components, only one of which poses any difficulty as it contains a log singularity. We are able to remove it as in Porter & Porter [2000] by applying Kummer's Transformation and then by integrating the log singularity analytically. The main attraction of this approach is that convergence in N is very rapid with typically only $N = 8$ at most required.

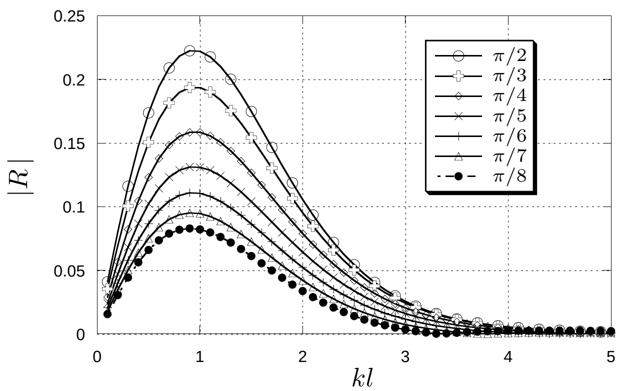

Figure 2: Oblique scattering for a range of angles of incidence over a topography defined by $\hat{h}(x) = \sin(\pi x/l)$, $h_{max} = h/2$, $l = h, N = 11$.

We stress that the example presented above is a specific example of an application of a more general technique that we have developed based on the construction of a vector Green's function as suggested by Noblesse[2004]. The Green's function is constructed such that $\nabla G = \nabla \times \mathbf{L}$ where $\mathbf{L} = (G_y^z, -G_x^z, 0)$ and Stokes' theorem is used to switch from normal to tangential derivatives. In several critical respects our approach differs from Noblesse in that we use a specially constructed Green's function and we apply this transformation twice to transform the integral equation into a form where highly accurate solution techniques may be employed. We present results above for oblique scattering by a submerged ridge, however, at the workshop we will also present results for scattering by an axisymetric topography. We are currently working on the oblique scattering problem for arbitrary topography joining two domains of constant but different depth. Concurrently we are also investigating the numerical approach to the problem of fully three dimensional scattering by an arbitrary patch of topography. We hope to be able to say more about these at the workshop.

References

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