Nonlinear harbor oscillations excited by random incident waves

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There are two types of forcing that excite long-period oscillations in a harbor. The first is the very long-period incident waves of tsunami origin. In this case a standing wave mode is excited synchronously by incident waves of the same frequency. The linearized long-wave theory and the weakly nonlinear improvement by Boussinesq approximation are pertinent, and effective numerical tools have been developed for applications. Resonance of long waves can also be forced nonlinearly by short-period waves of wind orgin. Though less dramatic in damages, wind-waves affect frequently all harbors in the world and can hinder the loading and unloading of cargos. While there are a few early studies focussed on narrow-banded incident waves Agnon & Mei [1]; Wu & Liu [7], incident sea spectra are nearly always broad-banded. In recent years, the Port of Long Beach has been plagued by such oscillations. Since the tools for prediction do not exist, physical models and mathematical models have been conducted by assuming linear synchronous resonance. New breakwaters have been built on the same basis, despite the knowledge that the sea spectrum outside the Port contains little long-wave energy. A theory for general bathymetry and harbor geometry, and random waves is long overdue.

In the linearized framework, the mild-slope equation (MSE) of Berkhoff [2] is a useful tool which reduces the computation of 3D refraction / diffraction problems to 2D, and can be efficiently solved numerically by, e.g., the hybrid element method of [4]. The original MSE has been extended by Chamberlain & Porter [3] to include both first- and second-order terms in the bed slope. In this paper two extensions of MSE are made. First, terms which are second order in nonlinearity is added. Second, the incident waves are random with prescribed frequency spectrum.

Specifically, at the first order in wave steepness $O(\epsilon = kA)$, the free surface of the incident wave is described by the Fourier integral

$$\zeta_1^{(I)}(r,\theta,t) = \int_{-\infty}^{\infty} A(\omega)\, e^{ik(\omega)r\cos(\theta-\theta_I)-i\omega t} d\omega, \tag{1}$$

where $A(\omega)$ is the random amplitude spectrum, $k(\omega)$ the wave number governed by the dispersion relation, and θ_I incident angle. The first-order velocity potential for the entire refraction-diffraction problem can be written as

$$\Phi_1(\mathbf{x},t) = \int_{-\infty}^{\infty} A(\omega)\, \phi_1(\mathbf{x},\omega)\, e^{-i\omega t} d\omega. \tag{2}$$

If the amplitude-normalized potential $\phi(\mathbf{x},\omega)$ is assumed to be of the form,

$$\phi_1(\mathbf{x},\omega) = -\frac{ig}{\omega}\frac{\cosh[k(\omega)(z+h)]}{\cosh[k(\omega)h]}\Gamma_1(x,y,\omega), \tag{3}$$

it is known that the transfer function Γ_1 is governed by the modified mild-slope approximation of Chamberlain & Porter [3],

$$\nabla \cdot \left[a_1 \nabla \Gamma_1 \left(x, y, \omega\right)\right] + \left[k^2 a_1 + g U_1 \nabla^2 h + g V_1 \left(\nabla h\right)^2\right] \Gamma_1 \left(x, y, \omega\right) = 0. \tag{4}$$

where a_1, U_1 and V_1 are known functions of kh. The elliptic boundary-value problem can be efficiently solved by the hybrid-element method, if the far field is modeled as a sea of constant depth, so that analytical representation can be used. Afterwards, the first-order surface elevation is given by

$$\zeta_1 \left(x, y, t\right) = \int_{-\infty}^{\infty} A\left(\omega\right) \Gamma_1 \left(x, y, \omega\right) e^{-i\omega t} d\omega. \tag{5}$$

At the second order in nonlinearity, the potential is expressed as a double Fourier integral,

$$\Phi_2 \left(\mathbf{x}, t\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A\left(\omega_1\right) A\left(\omega_2\right) \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right) e^{-i(\omega_1 + \omega_2)t} d\omega_1 d\omega_2, \tag{6}$$

where ϕ_2 must satisfy

$$\nabla^2 \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right) + \frac{\partial^2 \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right)}{\partial z^2} = 0, \quad -h(x, y) < z < 0, \tag{7}$$

$$\frac{\partial \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right)}{\partial z} = -\nabla \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right) \cdot \nabla h, \quad z = -h(x, y), \tag{8}$$

$$\frac{\partial \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right)}{\partial z} - \frac{\left(\omega_1 + \omega_2\right)^2}{g} \phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right) = f\left(x, y, \omega_1, \omega_2\right), \ z = 0. \tag{9}$$

The forcing term on the free surface is

$$\begin{aligned} f\left(x, y, \omega_1, \omega_2\right) = &\left[\frac{ig\left[k\left(\omega_2\right)\right]^2}{\omega_2} - \frac{i\omega_2^3}{g} - \frac{i\left(\omega_1 + \omega_2\right)\omega_1\omega_2}{g}\right] \Gamma_1 \left(x, y, \omega_1\right) \Gamma_1 \left(x, y, \omega_2\right) \\ &+ \left[-\frac{i\left(\omega_1 + \omega_2\right) g}{\omega_1 \omega_2}\right] \nabla_2 \Gamma_1 \left(x, y, \omega_1\right) \cdot \nabla_2 \Gamma_1 \left(x, y, \omega_2\right). \end{aligned} \tag{10}$$

Unlike the case of monochromatic incident waves, the transfer potential ϕ_2 involves two frequencies.

Denoting $\sigma = \omega_1 + \omega_2$, it is necessary to add evanescent modes for ϕ_2,

$$\phi_2 \left(\mathbf{x}, \omega_1, \omega_2\right) = -\frac{ig}{\sigma} \sum_{m=0}^{\infty} \xi_m \left(x, y, \omega_1, \omega_2\right) \frac{\cos \kappa_m (z + h)}{\cos \kappa_m h}, \tag{11}$$

where κ_m are the roots of the dispersion relation,

$$-\sigma^2 = g\kappa_m \tan \kappa_m h. \tag{12}$$

In particular, κ_0 is the imaginary root, corresponding to the propagating mode, while $\kappa_m, m = 1, 2, \ldots$ are the real roots, corresponding to the evanescent modes. By repeating the solvability argument based on Green's formula, we have found that $\xi_\ell\left(x,y,\omega_1,\omega_2\right)$ are governed by a matrix paritial differential equations:

$$\sum_{\ell=0}^{\infty}\left\{\nabla\cdot\left(A_{m\ell}\nabla\xi_\ell\right)+B_{m\ell}\nabla h\cdot\nabla\xi_\ell+C_{m\ell}\xi_\ell\right\}=-i\sigma f\left(x,y,\omega_1,\omega_2\right) \tag{13}$$

for both sum and difference frequencies. The left-side is the same as Porter & Staziker [5] for linear problems where $f=0$. It can be shown that the matrices on the left are diagonal and the equations uncoupled, only for constant depth. The coupled elliptic problem can again be solved by the hybrid-element method with the help of Green's function in the far field where the depth is constant.

The second-order free surface elevation can then be found from Bernoulli's theorem,

$$\zeta_2\left(\mathbf{x},t\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A\left(\omega_1\right)A\left(\omega_2\right)\Gamma_2\left(x,y,\omega_1,\omega_2\right)e^{-i(\omega_1+\omega_2)t}d\omega_1 d\omega_2, \tag{14}$$

with

$$\begin{aligned}\Gamma_2\left(x,y,\omega_1,\omega_2\right)=&\left(\frac{\omega_2^2}{g}+\frac{\omega_1\omega_2}{2g}\right)\Gamma_1\left(\omega_1\right)\Gamma_1\left(\omega_2\right)+\frac{g}{2\omega_1\omega_2}\nabla_2\Gamma_1\left(\omega_1\right)\cdot\nabla_2\Gamma_1\left(\omega_2\right)\\&+\sum_{\ell=0}^{\infty}\xi_\ell\left(\omega_1,\omega_2\right).\end{aligned} \tag{15}$$

Now we extend the nonlinear stochastic theory of Sclavounos [6] who studied the simpler problem of random wave reflection from a long vertical cliff near a sea of constant depth. Let $A(\omega)$ be a Gaussian random variable. Defining the covariance function by

$$H(\vec{x},\tau)=\overline{\zeta(\vec{x},t)\zeta^*(\vec{x},t+\tau)}. \tag{16}$$

Due to Gaussianity, ensemble averages of all odd products of random variables A and A^* vanish, and ensemble averages of all even products (e.g., quadratic) can be reduced to averages of quadratic products. It follows that

$$H(\tau)=\epsilon^2\overline{\zeta_1(t)\zeta_1^*(t+\tau)}+\epsilon^4[\overline{\zeta_2(t)\zeta_2^*(t+\tau)}+\overline{\zeta_1(t)\zeta_3^*(t+\tau)}+\overline{\zeta_3(t)\zeta_1^*(t+\tau)}]. \tag{17}$$

Note that nonlinear corrections start from $O(\epsilon^4)$ which requires in principle one to find not only the second-order response ζ_2 but the third-order response ζ_3. The last task is prohibitively complicated for a refraction/ diffraction problem.

The corresponding frequency spectrum is of the form

$$S(\omega)=\epsilon^2 S_2(\omega)+\epsilon^4 S_4(\omega) \tag{18}$$

where S_2 is the usual response spectrum in the linearized theory,

$$S_2=S_I(\omega)|\Gamma_1(\vec{x},\omega)|^2 \tag{19}$$

which depends only on the linear frequency response $\Gamma_1(\vec{x},\omega)$. The nonlinear spectral correction S_4 is composed of two parts

$$S_4=S_{22}+S_{13} \tag{20}$$

The first part is the self-product of second-order frequency response,

$$S_{22}(\omega) = \delta(\omega) \iint_{-\infty}^{\infty} S_I(\omega_1) S_I(\omega_2 - \omega_1) \Gamma_2(\omega_1, -\omega_1) \Gamma_2^*(\omega_2 - \omega_1, -\omega_2 + \omega_1) d\omega_1 d\omega_2$$
$$+ \int_{-\infty}^{\infty} S_I(\omega_1) S_I(\omega - \omega_1) \left\{ |\Gamma_2(\omega_1, \omega - \omega_1)|^2 + \Gamma_2(\omega_1, \omega - \omega_1) \Gamma_2^*(\omega - \omega_1, \omega_1) \right\} d\omega_1. \tag{21}$$

The second part

$$S_{13}(\omega) = 2 S_I(\omega) \Gamma_1(\omega) \int_{-\infty}^{\infty} S_I(\omega_1) \left\{ \Gamma_3^*(\omega, \omega_1, -\omega_1) + \Gamma_3^*(\omega_1, \omega, -\omega_1) + \Gamma_3^*(\omega_1, -\omega_1, \omega) \right\} d\omega_1 \tag{22}$$

depends on the third-order frequency response $\Gamma_3(\vec{x}; \omega_1, \omega, -\omega_1)$, in principle. Note however, that (i) the incident-wave spectrum $S_I(\omega)$ is outside the integral, and (ii) there is negligible energy at low frequencies in usual sea spectra, i.e., $S_I(\omega) \cong 0$ for small ω (see, e.g, JONSWAP). Since we are primarily interested in long-period response in the harbor, the part $S_{13}(\omega)$ is of negligible importance to $S_4(\omega)$. This fortunate result makes it unnecessary to compute Γ_3 and simplifies the task for the harbor problem. Note further that in Eq. (21), the second-order response function $\Gamma_2(\omega_1, \omega_2)$ needs to be computed only in a narrow strip ($\omega_1 + \omega_2 = \omega \ll 1$) of the frequency plane (ω_1, ω_2). Hence the numerical task is limited.

Numerical results for harbor response will be reported at the workshop.

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