Scattering of flexural waves by a pinned thin elastic sheet floating on water

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1 Introduction

In recent years a number of papers presented at the Workshops have been concerned with modelling the effect of waves on large floating structures where bending effects are important. Other papers have been concerned with the effect of waves on large numbers of cylindrical vertical rigid columns extending throughout the water depth. In both cases the motivation stemmed from the possibility of building massive off-shore runways for aircraft, either freely-floating or supported by many columns.

In this paper we continue in this vein by considering the water surface to be completely covered by a freely-floating thin elastic plate of uniform small thickness. It is well-known that the linearised equations describing the vertical displacement of the plate permit solutions describing long-crested flexural- gravity waves travelling through the plate and decaying with depth into the water region.

Here we ask what the effect of pinning the plate at any arbitrary number of points has on a given incident wave field. Of particular interest will be the forces required at each point to hold it fixed in the incident wave field, and the form of the scattered far field.

Before considering this problem the simpler problem of a thin elastic plate *in vacuo* pinned at given points is considered in the presence of an incident wave field. Although this problem is simpler, it would appear that few papers have addressed it despite the relevance to the vibrations of riveted sheets. See for example, Norris & Vermula (1995).

In both problems, extensive use is made of a Green function describing a concentrated force at a point on the plate. Because of the fourth-order differential operator describing the bending of the plate, both of these functions have bounded displacements at the point in question, which enables solutions for scattering by a incident field to be readily found by linear superposition of the Green functions.

In what follow, we shall present the theory for both cases but our results will concentrate on the simpler in *vacuo case*. Further results for the floating plate over water will be presented at the Workshop.

2 Formulation and solution

The plate occupies the $x-y$ plane and has a displacement $\mathbb{R}e\ u(\boldsymbol{r})\exp(-i\omega t)$ where ω is the radian frequency, and $\boldsymbol{r}=(x,y)$. Then it is known from Kirchhoff thin plate theory that $u(\boldsymbol{r})$ satisfies

$$(\nabla^4-k^4)u(\boldsymbol{r})=0, \qquad -\infty<x,y<\infty \tag{1}$$

where $\nabla^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$ where $k^4=m\omega^2/D$. Here $D=Eh^2/12(1-\nu^2)$ is the bending stiffness, E is Young's modulus, h the plate thickness and ν is Poisson's ratio.

A solution of (1) describing a long-crested plane wave making an angle ψ with the positive x-axis is

$$u_i(\boldsymbol{r})=\exp\{ik(x\cos\psi+y\sin\psi\}=\exp\{ikr\cos(\theta-\psi)\} \tag{2}$$

where $x=r\cos\theta, y=r\sin\theta$. A fundamental Green function satisfying

$$(\nabla^4-k^4)g(\boldsymbol{r};\boldsymbol{r'})=\delta(x-\xi)\delta(y-\eta)=\delta(\boldsymbol{r}-\boldsymbol{r'}) \tag{3}$$

where $\boldsymbol{r}=(x,y), \quad \boldsymbol{r'}=(\xi,\eta)$, is

$$g(\boldsymbol{r};\boldsymbol{r'})=C(H_0)^{(1)}(k\rho)-H_0^{(1)}(ik\rho), \quad C=i/8k^2 \tag{4}$$

where $\rho=|\boldsymbol{r}-\boldsymbol{r'}|$ and where $H_0^{(1)}$ is the Hankel function of order zero.A useful alternative formula is

$$\begin{aligned} g(\boldsymbol{r};\boldsymbol{r'}) &= C(\pi i)^{-1}\int_{-\infty}^{\infty}e^{ik(x-\xi)t} \\ &(\lambda^{-1}e^{-\lambda k|y-\eta|}-\gamma^{-1}e^{-\gamma k|y-\eta|})dt \end{aligned} \tag{5}$$

where

$$\lambda=(t^2-1)^{1/2}, t\geq 1, \quad \gamma=(t^2+1)^{1/2} \tag{6}$$

and $\lambda=-i\kappa$ for $t\leq 1$ where $\kappa=(1-t^2)^{1/2}$.

A crucial property of $g(\boldsymbol{r};\boldsymbol{r'})$ is that it is bounded as $\boldsymbol{r}\to\boldsymbol{r'}$. In fact

$$g(\boldsymbol{r};\boldsymbol{r'})\sim C+O(\rho\ln\rho), \rho\to 0. \tag{7}$$

Notice that this bounded behaviour for small r does not arise in the case of the fundamental Green function for the Helmholtz equation describing small acoustic vibrations given by the first term in (4), which is logarithmically singular as $\rho\to 0$. The boundedness described by (7) enables us to solve a number of interesting problems. For example, suppose the plate is pinned at the N points $\boldsymbol{r_n} = (x_n, y_n), n = 1, 2 \ldots N$ in the presence of the incident wave. Then the total displacement $u(\boldsymbol{r})$ is given by

$$u(\boldsymbol{r}) = u_i(\boldsymbol{r}) + \sum_{n=1}^{N} A_n g(\boldsymbol{r}; \boldsymbol{r_n}) \tag{8}$$

since (1) is satisfied everywhere except at the points $\boldsymbol{r_n}$ and the second term in (8) describes out-going waves as $\rho_n = |\boldsymbol{r} - \boldsymbol{r_n}| \to \infty$. Also, the requirement

$$u(\boldsymbol{r_m}) = 0, \quad m = 1, 2, \ldots N.$$

is satisfied if the A_n satisfy

$$\begin{aligned} \textstyle\sum_{n=1}^{\infty} \quad & A_n g(\boldsymbol{r_m}; \boldsymbol{r_n}) \\ & = -u_i(\boldsymbol{r_m}) = -\exp\{ikr_m \cos(\theta_m - \psi)\}, \\ & m = 1, 2, \ldots N. \end{aligned} \tag{9}$$

Again, suppose, in the absence of the incident wave, one of the points $\boldsymbol{r_m}$ is given a unit displacement, the rest remaining pinned. Then the displacement is given by

$$u_m(\boldsymbol{r}) = \sum_{n=1}^{N} B_{mn} g(\boldsymbol{r}; \boldsymbol{r_n}) \tag{10}$$

where the B_{mn} satisfy

$$\sum_{n=1}^{N} B_{mn} g(\boldsymbol{r_p}; \boldsymbol{r_n}) = \delta_{mp}, \quad p = 1, 2 \ldots N \tag{11}$$

It is straightforward to show that $B_{mn} = B_{nm}$ and

$$A_n = -\sum_{m=1}^{N} B_{mn} u_i(\boldsymbol{r_m}), \quad n = 1, 2 \ldots N \tag{12}$$

so that the general scattering problem may be expressed in terms of the solution to N distinct radiation problems.

Returning to the scattering problem it is possible to determine information about the far-field scattered waves by using the result

$$g(\boldsymbol{r}; \boldsymbol{r_n}) \sim f_n(\theta) \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp\left\{i\left(kr - \frac{\pi}{4}\right)\right\}, \; r \to \infty \tag{13}$$

where

$$\begin{aligned} f_n(\theta) \quad & = C\exp\{-ik(x_n \cos\theta + y_n \sin\theta)\} \\ & = C\exp\{-ikr_n \cos(\theta - \theta_n)\} \end{aligned} \tag{14}$$

and (r_n, θ_n) are the polar co-ordinates of (x_n, y_n). It follows from (8) that the scattered wave $u_{sc}(\boldsymbol{r})$ satisfies

$$u_{sc}(\boldsymbol{r}) \sim D(\theta, \psi) \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp\{i(kr - \pi/4)\} \quad r \to \infty \tag{15}$$

where

$$D(\theta, \psi) = C \sum_{n=1}^{N} A_n e^{-ikr_n \cos(\theta - \theta_n)} \tag{16}$$

in a notation used in Norris & Wang (1994).

For a given distribution of pinned points $\boldsymbol{r_n}, \; n = 1, 2, \ldots N$, it is a simple matter to compute the $N \times N$ system (9) for the complex force coefficients A_n at each point which are required to hold the points at rest, and to compute the amplitude and phase of the diffraction coefficient $D(\theta, \psi)$ in a direction θ, for an incident wave direction ψ from (33).

Figure 1 shows the normalised value of $|A_n|$ ($|A_n|/8k^2$). The first and second plots are for grids of 16 and 25 points respectively spaced ka apart while the third and fourth plots are for a grid of 16 and 25 points spaced ka apart plus a small random amount. The random amount added was choosen from a normal distribution with mean zero and standard deviation $ka/10$. It is clear from this figure that the coeffient $|A_n|$ (which may be thought of as the force) has significant spikes for certain values of ka for symmetric grids but that these spikes are lost if the symmetry is broken (by the addition of randomness). Similar results for a circle are shown in figure 2. The results are similar to those observed in Evans & Porter (1999)

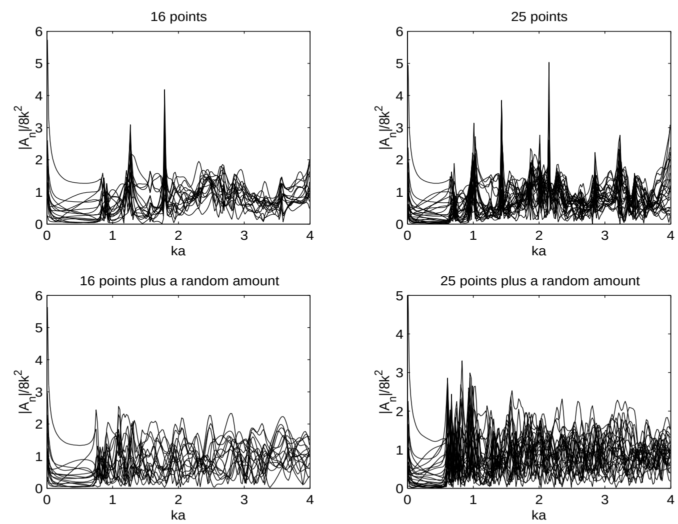

Fig. 1: The normalised coefficient $|A_n|$ for grids of 16 and 25 points evenly spaced ka apart or spaced ka apart plus a small random amount. The incident angle is $-\pi/3$.

3 A floating elastic plate over water

No major conceptual difficulty arises in considering a thin elastic plate floating on water of depth h, which again is pinned at points $\boldsymbol{r_n}, \; n = 1, 2 \ldots N$. The equation for the displacement $u(x, y)$ is (1) as before but now there is a pressure term on the right-hand-side due to the water. Thus, $u(\boldsymbol{r})$ satisfies

$$(D\nabla^4 - m\omega^2)u(\boldsymbol{r}) = p(\boldsymbol{r}) \tag{17}$$

where

$$p(\boldsymbol{r}) = i\rho\omega\phi - \rho g u \tag{18}$$

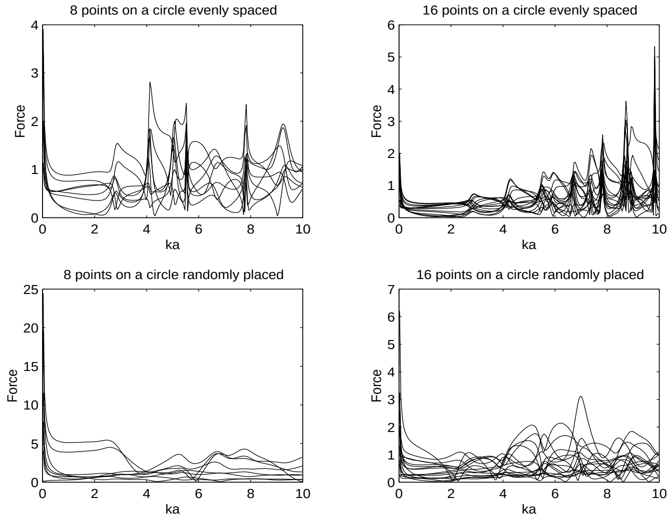

Fig. 2: The normalised coefficient $|A_n|$ for 8 or 16 points on a circle of radius ka evenly spaced or randomly choosen. The incident angle is $-\pi/3$.

from Bernoulli's equation, where ρ is the water density and

$$\phi(\boldsymbol{r}, z) = \mathbb{R}e\, \Phi(\boldsymbol{r}, z) e^{-i\omega t} \tag{19}$$

is a harmonic time-independent velocity potential for the fluid motion, satisfying

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) \phi(\boldsymbol{r}, z) = 0 \tag{20}$$

in the water region and the no-flow condition

$$\phi_z = 0,\ z = -h. \tag{21}$$

Elimination of p, u from (17), (18) using the relation

$$-i\omega u = \phi_z, \quad z = 0 \tag{22}$$

gives the set of conditions to be satisfied by ϕ, namely (20), (21) and

$$(\beta_0 \nabla^4 + 1 - \delta)\phi_z - \mu\phi = 0,\ z = 0 \tag{23}$$

where $\beta_0 = D/\rho g,\ \delta = m\omega^2/\rho g,\ \mu = \omega^2/g$.

Solutions of (20),(21), (23) are

$$\exp\{ik(x\cos\psi + y\sin\psi)\}\cosh k(z+h) \tag{24}$$

provided k satisfies

$$K(k) = (\beta_0 k^4 + 1 - \delta)k\sinh kh - \mu\cosh kh = 0. \tag{25}$$

This has roots $\pm k_n (n = -2, -1, 0, 1, 2, \dots)$ where k_0 is real and positive, k_{-1}, k_{-2} are complex with imaginary parts positive, and $k_n, n \geq 1$ are pure imaginary with positive imaginary part.

Thus a flexural wave given by the potential

$$\phi_i(\boldsymbol{r}, z) = \exp\{ik_0(x\cos\psi + y\sin\psi)\}\cosh k_0(z+h) \tag{26}$$

describes a wave making an angle ψ with the positive x-axis.

Then the incident wave displacement is

$$u_i(\boldsymbol{r}) = A\exp\{ik_0(x\cos\psi + y\sin\psi)\} \tag{27}$$

where

$$A = i\omega^{-1}k_0\sinh k_0 h \tag{28}$$

The solution describing the scattering of this incident wave by pinned points $\boldsymbol{r_n}$ now proceeds as before. Thus the total potential is

$$\phi(\boldsymbol{r}, z) = \phi_i(\boldsymbol{r}, z) + \sum_{n=1}^{N} C_n G(\boldsymbol{r}, z; r_n) \tag{29}$$

where the C_n satisfy

$$\sum_{n=1}^{N} C_n G_z(\boldsymbol{r_m}, 0; r_n) = -\phi_{iz}(r_m, 0) \qquad m = 1, 2..N \tag{30}$$

Here $G(\boldsymbol{r}, z; \boldsymbol{r'})$ satisfies (20), (21) and (23) with right-hand-side replaced by $\delta(\boldsymbol{r} - \boldsymbol{r'})$, and is outgoing for large r.

This Green function has been derived by Fox & Chung (2002) and may be written

$$G(\boldsymbol{r}, z; \boldsymbol{r'}) = \frac{i}{2}\sum_{n=-2}^{\infty} \frac{\cosh k_n(z+h)}{K'(k_n)} k_n H_0^{(1)}(k_n\rho). \tag{31}$$

4 Infinite Array of points

Suppose we consider the scattering of the incident wave given by an infinite periodic set of pinned points $(na, 0), n \in \mathbb{Z}$. Then periodicity dictates that $A_n = e^{i\beta na}A_0$ so that 14 becomes

$$A_0 \sum_{n=-\infty}^{\infty} e^{i\beta na} g(ma, 0; na, 0) = -e^{i\beta ma} \quad m \in \mathbb{Z} \tag{32}$$

and the solution becomes

$$u(x, y) = u_i(x, y) + A_0 \sum_{n=-\infty}^{\infty} e^{i\beta na} g(x, y; na, 0) \tag{33}$$

Equation (32) simplifies to

$$-A_0^{-1} = \sum_{n=-\infty}^{\infty} e^{-i\beta na} g(na) \tag{34}$$

after redefining the summation index and making use of properties of g. Here $g(na) \equiv g(na, 0; 0, 0)$. We can use Poisson's formula

$$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{inu} F(u) du = 2\pi \sum_{n=-\infty}^{\infty} F(2n\pi) \tag{35}$$

to express A_0 in the form

$$A_0 = -4k^3 a S^{-1}(\psi) \tag{36}$$

where

$$S(\psi)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{\lambda_n}-\frac{1}{\gamma_n}\right) \tag{37}$$

and $\lambda_n=((\beta_n/k)^2-1)^{\frac{1}{2}}, \quad \gamma_n=((\beta_n/k)^2+1)^{\frac{1}{2}}$,

$$\beta_n=\beta+2n\pi/a \tag{38}$$

It is convenient to define scattering angles $\psi_n, n\in\mathbb{Z}$ by

$$\beta_n=k\cos\psi_n \tag{39}$$

where $\beta_0=\beta=k\cos\psi_0=k\cos\psi$. Then provided $|\beta_n|\leq k$ or

$$|\cos\psi_n|=|\beta/k+2n\pi/ka|\leq 1 \tag{40}$$

the ψ_n are real with $0\leq\psi_n\leq\pi$ and $\lambda_n=-i\sin\psi_n$. In this case we say $n\in\mathcal{N}$. It is clear that the set $\mathcal{N}$ always has at least one member $n=0$ corresponding to $-\psi_0=-\psi$ the incident wave angle. For $n>0$, the number of scattering angles ψ_n increases with increasing ka according to (40). We have

$$\begin{aligned} S(\psi) &= \sum_{n\in\mathcal{N}}\left(\frac{i}{\sin\psi_n}-\frac{1}{(1+\cos^2\psi_n)^{\frac{1}{2}}}\right) \\ &+\sum_{n=-\infty,\ n\notin\mathcal{N}}^{\infty}\left(\frac{1}{\lambda_n}-\frac{1}{\gamma_n}\right)\end{aligned} \tag{41}$$

and the infinite sum converges absolutely since

$$\lambda_n^{-1}-\gamma_n^{-1}\sim(ka/2\pi n)^3 \quad \text{as} \quad n\to\infty. \tag{42}$$

Also the finite sum exists provided $\sin\psi_m\neq 1$ for some m, or $\psi_m=0,\pi$ where $\cos\psi_m=\cos\psi+2m\pi/ka$. This situation is termed resonance by Hills & Karp (1965).

To see the effect on the scattered field we note from (33) that this is

$$u_{sc}(x,y)=A_0\sum_{n=-\infty}^{\infty}e^{i\beta na}g(x,y;na,0) \tag{43}$$

$$=-\sum_{n=-\infty}^{\infty}e^{i\beta_n x}\left(\frac{e^{-\lambda_n k|y|}}{\lambda_n}-\frac{e^{-\gamma_n k|y|}}{\gamma_n}\right)/S(\psi)$$

where Poisson's formula (35) has been used again. It is clear from (43) and (37) that $u_{sc}(ma,0)=-1, m\in\mathbb{Z}$ as expected.

The scattered field involves plane waves arising from $n\in\mathcal{N}$ in (43) only. Thus

$$u_{sc}(x,y)\sim -iS^{-1}(\psi)\sum_{n\in N}\frac{e^{ikr\cos(\vartheta-\mathrm{sgn}y\psi_n)}}{\sin\psi_n} \tag{44}$$

where $x=r\cos\vartheta, \quad y=r\sin\vartheta$, which describes a number of plane waves each making angles ψ_n with the positive x-axis.

The amplitude of each wave is $(iS(\psi)\sin\psi_n)^{-1}$. If however $\psi_n=0,\pi$ for a value $n=m$, say, then $S(\psi)\sim i/\sin\psi_n$ and $u_{sc}(x,y)\sim -e^{ikx\cos\psi_n}$ which describes a wave propagating along the x-axis either towards $x=\infty$ if $\psi_n=0$ or towards $x=-\infty$ if $\psi_n=\pi$. If ka is small enough, there is only a single scattered wave

$$\begin{aligned} u_{sc}(x,y) &\sim -iS^{-1}(\psi)e^{ikx\cos(\vartheta-\mathrm{sgny}\psi_0)}/\sin\psi_0 \\ &\sim \frac{-iS^{-1}(\psi)}{\sin\psi_0}(e^{ikx\cos\psi_0\pm iky\sin\psi_0}) \qquad y\lessgtr 0.\end{aligned} \tag{45}$$

Thus $R_0=-iS^{-1}/\sin\psi_0 \quad T_0=1-iS^{-1}/\sin\psi_0$.

Figure 3 shows the real part of the displacement u for a wave incident for angles of $-\pi/4$ and $-\pi/2$ on an array of points spaced $ka=1.5\pi$ and $ka=3\pi$ apart. For the case $ka=1.5\pi$ there is a single scattered wave while for $ka=3\pi$ we have three scattered waves and this can be seen in the results especially for the case when the wave incident is at $-\pi/2$.

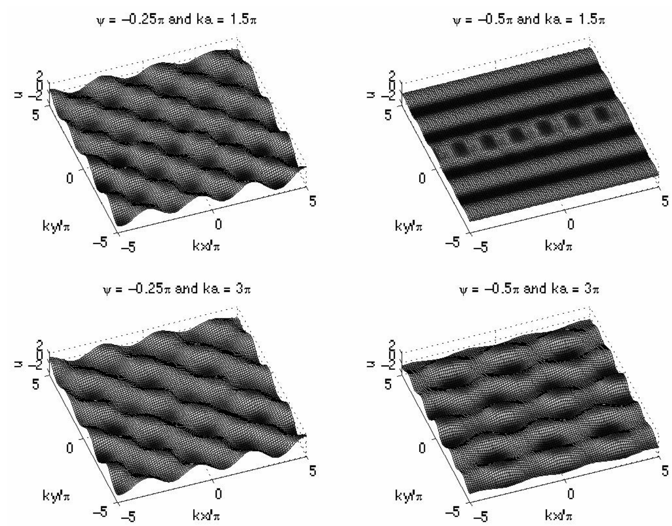

Fig. 3: The real part of the displacement for a wave incident on an array for the incident angles ψ and spacings ka shown.

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