Fully nonlinear and simplified models for 3D water waves generated by a moving bottom

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We study waves interacting with a geometry or wave motion caused by the motion of a geometry. In the present study, the geometry is represented by an uneven bottom of arbitrary shape in space and time. The formulation is three dimensional and fully nonlinear. Various versions of the scheme are tested out. Some parts of the scheme are brought into analytical form which is suitable for analysis and numerical tests.

We consider irrotational wave motion at the surface of a homogeneous incompressible fluid over an impermeable bottom. $\mathbf{x}=(x,y)$, z, denotes horizontal and vertical coordinates. $z=0$, $z=\beta(\boldsymbol{x},t)$ and $z=\eta(\mathbf{x},t)$ are, respectively, the equations of the still level, of the impermeable bottom and of the free surface.

We introduce the potential function $\tilde{\phi}(\mathbf{x},t)$ at the free surface determined by $\tilde{\phi}(\mathbf{x},t)=\phi(\mathbf{x},\eta(\mathbf{x},t),t)$. The kinematic and dynamic boundary conditions at $y=\eta(\mathbf{x},t)$ give

$$\eta_t - V_s = 0, \tag{1}$$

$$\tilde{\phi}_t + g\eta + \frac{|\nabla\tilde{\phi}|^2 - {V_s}^2 - 2V_s\,\nabla\eta\cdot\nabla\tilde{\phi} + |\nabla\eta\times\nabla\tilde{\phi}|^2}{2(1+|\nabla\eta|^2)} = 0, \tag{2}$$

where, $V_s=\phi_n\sqrt{1+|\nabla\eta|^2}$ and ϕ_n denotes the normal velocity and ∇ the horizontal gradient.

The Laplace equation (resulting from incompressibility and irrotationality), together with the bottom impermeability, is solved exactly by means of a Green function. After some algebra, this leads to the formulation:

$$\begin{aligned}
\mathfrak{F}(V_s) \;=\; & \mathfrak{F}(V_0) - k\tanh(kh)\,\mathfrak{F}\{\eta V_0\} - \mathrm{i}\boldsymbol{k}\cdot\mathfrak{F}\{\eta\nabla\tilde{\phi}\} \\
& + k/(1+e_h)[e_h\mathfrak{F}(\eta(V_s-V_0)) + \mathfrak{F}(\eta\mathfrak{F}^{-1}[e_h\mathfrak{F}(V_s-V_0)])] \\
& + k/(1+e_h)[\mathfrak{F}(T(\tilde{\phi})+T_1(\tilde{\phi})) + \mathfrak{F}(N(V_s)+N_1(V_s))] \\
& + k/(1+e_h)[2\sqrt{e_h}/k\mathfrak{F}(V_b) - 2\mathfrak{F}(\eta\mathfrak{F}^{-1}[\sqrt{e_h}\mathfrak{F}(V_b)]) - \mathfrak{F}(N^B(V_b)] \\
& + k/(1+e_h)[2\mathrm{i}\sqrt{e_h}/k\boldsymbol{k}\cdot\mathfrak{F}\{(\beta+h)\nabla\phi^b\} - \mathfrak{F}(T^B(\phi^b)]
\end{aligned} \tag{3}$$

where $V_0 = \mathfrak{F}^{-1}[\,k\tanh(kh)\,\mathfrak{F}(\tilde{\phi})]$ and $e_h \equiv \mathrm{e}^{-2kh}$. Here $V_b{=}\partial\phi/\partial n\sqrt{1{+}|\nabla\beta|^2}$, $\vec{n}$ being either $\vec{n_b}$, the inward normal of unit length at the bottom surface or $\vec{n_s}$, the outward normal of unit length at the free surface as illustrated in figure 1.

The kernels of the inner integrals T, T_1, N, N_1, T^B and N^B are quickly decaying in space. These integrals are evaluated numerically over a very limited region of the $\mathbf{x}$-plane.

If we now consider the case where the bottom is steady, i.e. $\beta(\boldsymbol{x},t) = \beta(\boldsymbol{x})$. The bottom surface beeing considered material, the cinematical condition leads to $V_b = \partial\beta/\partial t$ beeing identically null. In consequence the fourth line in equation (3) vanishes. If moreover the bottom is considered flat, $\beta(\boldsymbol{x},t) = -h$, the fourth and fifth lines in equation (3) vanish and one recover the results obtained in Fructus *et al.* (2005).

Providing now V_b given, one need to compute ϕ^b when $\eta, \tilde{\phi}, V_s$ are known. To achieve that, one can write the second Green's identity at a point of the bottom surface. This leads to the following expression for ϕ^b:

$$\begin{aligned}\mathfrak{F}(\phi^b) \;=\; & \mathfrak{F}(\phi^b_1) + 2\mathrm{i}\boldsymbol{k}/k\cdot(\sqrt{e_h}\mathfrak{F}\{\eta\nabla\tilde{\phi}\} - \mathfrak{F}\{(\beta+h)\nabla\phi^b_1\}) - 2\sqrt{e_h}\mathfrak{F}(\eta V_s) \\ & +[\mathfrak{F}(M^B(V_b) + H^B(\tilde{\phi})) + \mathfrak{F}(S^B(V_s))] \\ & -2\mathrm{i}\boldsymbol{k}/k\cdot\mathfrak{F}\{(\beta+h)\nabla(\phi^b-\phi^b_1)\} + \mathfrak{F}(D^B(\phi^b)] \end{aligned} \tag{4}$$

where $\phi^b_1 = 2\mathfrak{F}^{-1}[\,\sqrt{e_h}\mathfrak{F}(\tilde{\phi}) + 1/k(\sqrt{e_h}\mathfrak{F}(V_s) - \mathfrak{F}(V_b))]$.

As before, The kernels of the inner integrals M^B, H^B, S^B and D^B are quickly decaying in space. These integrals are also evaluated numerically over a very limited region of the $\mathbf{x}$-plane.

The system of equations {(3,4)} as to be solved iterativelly. In practice, few iterations are required, however.

In addition to the fully nonlinear modelling, we develop several simplified models. The integrals in equations 3 and 4 are expanded in terms of convolution sums plus some reminders. The highest order the convolutions considered, the quickiest the kernel of the reminders is decaying in space (see Fructus *et al.* (2005) for more details about this expansion). The considered extra convolutions can then be expressed analytically in closed form via Fourier transform. The higher order reminders are then neglected. This allow us to express an explicit solution for V_s for any given order.

If only the linear terms are considered, this leads to the linear solution:

$$V_1 \;= V_0 + \mathfrak{F}^{-1}(\frac{2\sqrt{e_h}}{1+e_h}\mathfrak{F}(V_b)) \tag{5}$$

If the expansion of the integrals is carried out by considering all convolutions of order two, this leads to the quadratic explicit formulation for the operator V_s:

$$V_2 = V_1 - \mathfrak{F}^{-1}[\,k\tanh(kh)\mathfrak{F}(\eta V_1) - \mathrm{i}\boldsymbol{k}\cdot\mathfrak{F}\{\eta\nabla\tilde{\phi}\} + \frac{2\sqrt{e_h}}{1+e_h}\mathrm{i}\boldsymbol{k}\cdot\mathfrak{F}\{(\beta+h)\nabla\phi^b_1\}] \tag{6}$$

Similarly for the potential at the bottom, one obtain:

$$\begin{aligned}\phi^b_2 = \phi^b_1 & - \frac{2}{k}\mathfrak{F}^{-1}[\sqrt{e_h}k\mathfrak{F}(\eta V_1) + \mathrm{i}\boldsymbol{k}\cdot\mathfrak{F}\{(\beta+h)\nabla\phi^b_1\}] \\ & + \frac{2}{k}\mathfrak{F}^{-1}[\sqrt{e_h}\mathfrak{F}(V_2-V_1) + \sqrt{e_h}\mathrm{i}\boldsymbol{k}\cdot\mathfrak{F}\{\eta\nabla\tilde{\phi}\}] \end{aligned} \tag{7}$$

Finally, if all cubic convolutions are considered as well, this leads to the following explicit cubic expression for the operator:

$$\begin{aligned} V_3 = V_2 - \mathfrak{F}^{-1}[\,k\tanh(kh)\mathfrak{F}(\eta(V_2 - V_1))] - \mathfrak{F}^{-1}[\,\frac{k^2}{2}\mathfrak{F}(\eta^2 V_1)] \\ +\mathfrak{F}^{-1}[\frac{k}{1+e_h}\left\{\mathfrak{F}(\eta\mathfrak{F}^{-1}[-\mathrm{i}\boldsymbol{k}\cdot\mathfrak{F}\{\eta\nabla\tilde{\phi}\}]) + \mathrm{i}\frac{e_h}{2}\boldsymbol{k}\cdot\mathfrak{F}\{\eta^2\nabla\tilde{\phi}\} - \mathfrak{F}\{\eta^2\mathfrak{F}^{-1}[\frac{k^2}{2}\tilde{\phi}]\}\right\}] \\ +\mathfrak{F}^{-1}[\frac{k}{1+e_h}\left\{k\sqrt{e_h}\mathfrak{F}\{(\beta+h)^2 V_b\} + 2\mathrm{i}\frac{\sqrt{e_h}}{k}\boldsymbol{k}\cdot\mathfrak{F}\{(\beta+h)\nabla(\phi_2^b - \phi_1^b)\}\right\}] \end{aligned} \quad (8)$$

This last expression is a kind of Zakharov equation with the effect of a moving bottom in space and time added. The formulation is explicit and analytic in three dimensions. Resulting computations using the formulae 8 are then straightforward and requires only FFT computations, which is very fast.

The validity of those models is then checked by comparison with the fully nonlinear scheme. Such results for a two dimentional case are presented in the figures 2, 3 and 4. The figure 2 presents the bottom variation in space and time, while the figures 3 and 4 display the corresponding wave elevation computed with the four formulations.

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References

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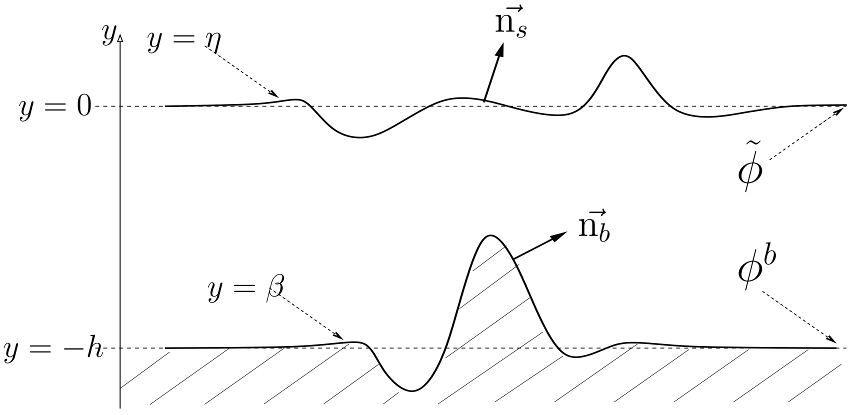

Figure 1: Sketch of the model

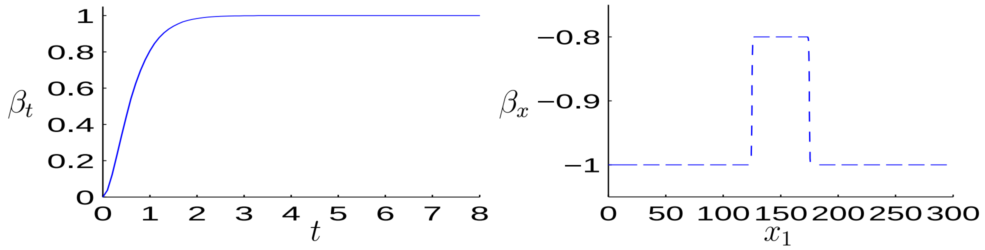

Figure 2: Time and spatial dependence β_t and β_x for the bottom elevation where $\beta(\boldsymbol{x}, t) = \beta_t(t)\beta_x(\boldsymbol{x})$.

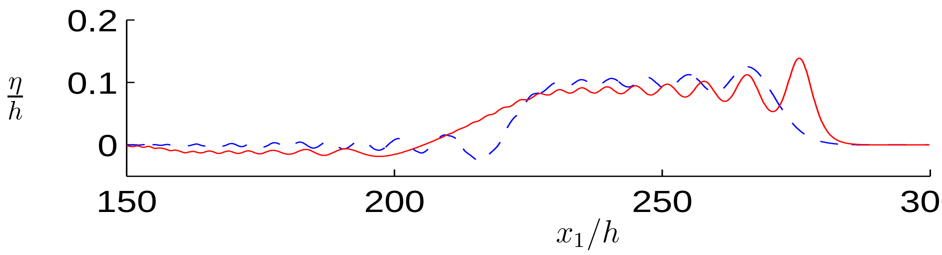

Figure 3: Surface elevation due to a sudden rise of the bottom (as defined in figure 2). surface at $t = 100\sqrt{h/g}$, fully nonlinear (—) and linear solution (- -).

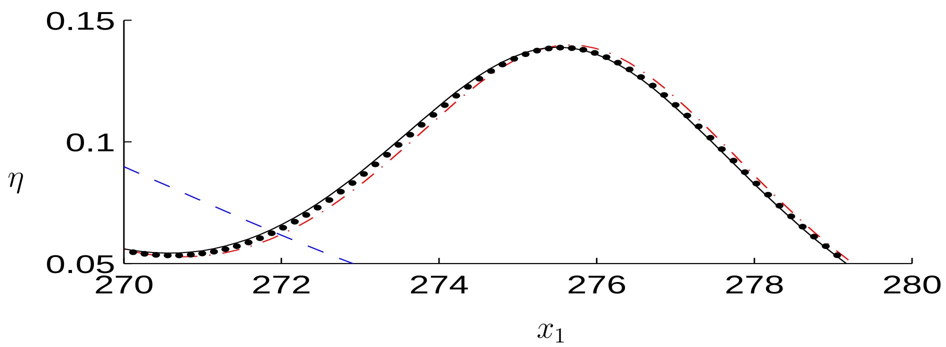

Figure 4: Blow-up of figure 3 at $t = 100\sqrt{h/g}$.
Computations based on V_2 (- · -), based on V_3 (· · ·), fully nonlinear (—) and linear solution (- -).