Three-dimensional Wagner problem using variational inequalities

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Introduction

Water impact remains a challenge for researchers. Wagner introduced in 1932 a simplified model of great use [12]. A main issue of the Wagner model is that the contact line between the disturbed free surface of the liquid and the surface of the entering body is unknown in advance and is a part of the solution. For two-dimensional and axisymmetric cases, the Wagner problem has been extensively studied in the past. Efficient methods for these two cases have been developed. In three-dimensional case, the contact line is a two-dimensional unknown curve. During the last decade the three-dimensional Wagner problem received increasing attention, leading to several analytical solutions and novel numerical algorithms.
In this paper the linearized Wagner problem is solved by using the variational inequality method [4], which finally reduces the original problem to a constrained minimization problem. The obtained results are compared with known analytical solutions. A fairly good agreement of the numerical results with the exact solutions is obtained. The developed method deals with unknown distribution of the displacement potential over a finite part of the boundary of the flow domain and does not require meshing the whole flow domain. The latter is important to reduce the computational time for 3D impact problems. Linear finite elements and an adaptive re-meshing algorithm are used to evaluate accurately both the displacement potential on the wetted part of the entering body surface and the shape of this wetted part.

Variational inequality for the Wagner problem

We consider the three-dimensional problem of unsteady liquid flow arising when a blunt body enters an ideal incompressible liquid through its free surface. At time $t = 0$, the liquid is at rest and occupies the lower half-space, $\{z \leqslant 0\} = \Omega = \{(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^-\}$; a blunt body touches the liquid free surface, $\{z = 0\} = \{(x, y, z) \in \mathbb{R}^2 \times \{0\}\}$ at a single point taken as the origin of the Cartesian coordinate system $(Oxyz)$. The liquid flow caused by the impact is assumed potential.

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Within the Wagner approach the boundary conditions on both the liquid free surface $FS(t)$ and the wetted part $WS(t)$ of the entering body surface are linearized and imposed on the initial position of the liquid surface, $\{z = 0\}$. The line $\Gamma(t)$, which separates the regions $FS(t)$ and $WS(t)$, is unknown and has to be determined as a part of the solution.
The arising boundary-value problem can be reduced to a variational inequality (see [4]) by using the concept of the displacement potential ϕ^D defined as:

$$\phi^D(\vec{x}, t) = \int_0^t \phi(\vec{x}, \tau)\, d\tau \tag{1}$$

where ϕ is the velocity potential. This approach was reproduced in [6], two-dimensional numerical calculations were performed in [3] and first three-dimensional numerical computations in [10].
In this section, the main steps of the approach based on the formulation of the water impact problem as a variational inequality are outlined.
The boundary value problem for the displacement potential has the form (see [4]):

$$\begin{cases} \Delta\phi^D = 0 & \text{in } \Omega = \{z \leqslant 0\} \\ \phi^D = 0 & \text{on } FS(t) \\ \dfrac{\partial \phi^D}{\partial z} = f(x, y) - h(t) & \text{on } WS(t) \\ \phi^D \to 0 & \text{when } x^2 + y^2 + z^2 \to \infty \end{cases} \tag{2}$$

where the function $f(x, y)$ describes the body shape and $h(t)$ is the penetration depth, $h(0) = 0$, $f(0, 0) = 0$. Pressure $p(x, y, t)$ is calculated by using the linearized Bernoulli equation $p = -\rho_w \phi^D_{,tt}$, where ρ_w is the liquid density. The contact line $\Gamma(t)$ in the boundary-value problem (2) is determined from the condition that the displacements of the liquid particles, $\nabla\phi^D$, are finite. It was shown in [4] that the latter condition is equivalent to the classical Wagner condition but much easier to deal with.
Within the approach derived in [4] two extra conditions are used. The first condition follows from the requirement that the pressure is positive in the contact region $WS(t)$ and the second condition that the liquid particles cannot penetrate the body surface. By integrating the inequality $p \geqslant 0$ twice in time, we obtain:

$$\phi^D \leqslant 0 \text{ on } WS(t). \tag{3}$$

The second condition provides:

$$\frac{\partial \phi^D}{\partial z} \leqslant f(\vec{x}) - h(t) \text{ on } \{z = 0\}. \tag{4}$$

Taking into account the boundary conditions in (2), we obtain the equality:

$$\phi^D\left(\frac{\partial\phi^D}{\partial t}+h(t)-f(\vec{x})\right)=0 \text{ on } \{z=0\}. \quad (5)$$

We introduce the symmetric bilinear form a and the linear form l defined by:

$$a(u,v)=\iiint_{\{z\leqslant 0\}}\nabla u\nabla v\, dx\, dy\, dz, \quad (6)$$

$$l(v)=\iint_{\{z=0\}}(f(\vec{x})-h(t))\, v\, dx\, dy, \quad (7)$$

and the functional space $W^1(\Omega)$ (see [7])

$$W^1(\Omega)=\left\{v\ ;\ \frac{v}{\sqrt{1+|\vec{x}|^2}},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial z}\in L^2(\Omega)\right\}, \quad (8)$$

where $L^2(\Omega)$ is the set of functions which are square integrable on Ω, and the cone $K\subset W^1(\Omega)$ is the set of elements of $W^1(\Omega)$, which are negative or zero on $\{z=0\}$. One can show that the problem for the displacement potential with the boundary inequalities (3), (4) and the boundary condition (5) can be reduced to a variational inequality:

$$a(\phi^D,v-\phi^D)\geqslant l(v-\phi^D),\forall v\in K. \quad (9)$$

It is proved in [7] that the bilinear form a is coercive on functional space $W^1(\Omega)$. This property allows us to say (see [1]) that (9) has a unique solution in K and that this solution minimizes on K the functional:

$$J(v)=\frac{1}{2}a(v,v)-l(v) \quad (10)$$

Thus the variational inequality problem can be reduced to the constrained minimization of J on the cone K. See for example [11] for more details on the methods used to solve this kind of problem.

Bilinear form a on $\{z=0\}$

Equations (6) and (7) define the symmetric bilinear form a and the linear form l. Since it is impossible to mesh the whole *infinite* fluid domain $\{z\leqslant 0\}$ and since we do not want to impose fixed and arbitrary boundaries in the far field, the bilinear form a is represented in another form, using the fact that the final solution must satisfy the Laplace equation in the flow domain. This is equivalent to searching for the function v minimizing J in the subset of K which contains only harmonic functions. Moreover, it is possible to replace the triple integral in (6) by a two dimensional integral over a finite region *without new approximation.*

We know that $\phi^D=0$ on the free surface. It is possible to build a closed domain D of $\{z=0\}$, such that one is *sure* that for every point of $\{z=0\}$ and outside of D, $\phi^D=0$ (see figure 1).

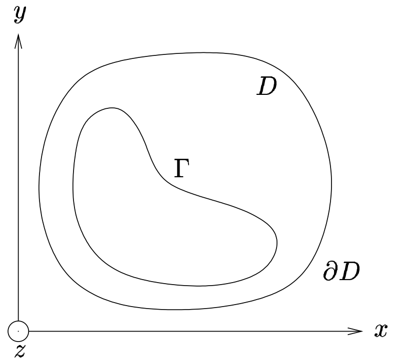

Figure 1: Rewriting bilinear form a on a finite domain of {z=0}

By using the known solution of the Dirichlet problem for the lower half-space, we obtain:

$$a(u,v)=\frac{1}{2\pi}\iint_{(x,y)\in D}\iint_{(x_0,y_0)\in D}\nabla_2 u(x,y)\nabla_2 v(x_0,y_0)\frac{dx_0\, dy_0\, dx\, dy}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \quad (11)$$

where $\nabla_2=(\frac{\partial}{\partial x},\frac{\partial}{\partial y})$ is the planar gradient on the liquid surface, $\{z=0\}$. Linear finite elements are used to approximate the integrals in (11).

First the domain D is decomposed into a set of elementary panels and the gradients $\nabla_2 u$ and $\nabla_2 v$ are approximated by constants vectors on each panel. Then the bilinear form a in (6) can be approximated by:

$$a(v,v)\approx\vec{v}\cdot A\cdot\vec{v}, \quad (12)$$

where $\vec{v}$ is the vector, components of which, v_i, are unknown values of the displacement potential at the nodes of the panels, and A is a symmetric matrix. Components of the matrix A were obtained analytically in the 2D case and are calculated numerically in the 3D case. In the same way the linear form l in (7) is approximated as

$$l(v)\approx\vec{L}\cdot\vec{v}, \quad (13)$$

where the vector $\vec{L}$ is dependent on both the entering body shape and the penetration depth. This vector is evaluated numerically.

Finally the original problem is reduced to the problem of minimizing the following quadratic form under the constraint that all the elements v_i of the vector $\vec{v}$ are negative or zero:

$$\min_{v_i\leqslant 0}\left(\frac{1}{2}\vec{v}\cdot A\cdot\vec{v}-\vec{L}\cdot\vec{v}\right) \quad (14)$$

Numerical results

The aim of this section is to compare the obtained numerical results with the analytical solutions. The three-dimensional method is tested with the help of the analytical solution for a cone [8]. The inverse Wagner theory

[9] also leads to analytical solutions, which are used in this section. Finally, numerical computation for the pyramid impact problem, which up to now has no analytical solution to our knowledge, are performed.
In order to optimize computation time with respect to precision of the result, we use an adaptive re-meshing algorithm (see [2]).
In cone and pyramid entry problems the flow is self-similar within the Wagner approach and depends on the deadrise angle in a simple way. Once numerical calculations have been performed for a particular deadrise angle, the numerical results can be applied to any deadrise angle by using appropriate stretching of the spacial coordinates and the displacement potential.

Cone entry problem

The axisymmetric solution of the Wagner problem is known (see [8]). Let β be the deadrise angle of the cone. We denote $r = \sqrt{x^2+y^2}$, $r_W(t)$ the radius of the circular contact line, $\rho = r/r_W$ and $h(t)$ the penetration depth. The radius $r_W(t)$ is given as:

$$r_W = \frac{4h}{\pi \, \tan\beta}, \qquad (15)$$

and the displacement potential in the contact region as:

$$\phi^D(\rho) = \frac{1}{4} r_W^2 \tan\beta \left(\rho^2 \,\mathrm{Argch}\, \frac{1}{\rho} - \sqrt{1-\rho^2} \right) \qquad (16)$$

We chose to test the variational inequality method with $h = 0.4$ and $\tan\beta = 1.5$. Figure 2 presents the difference between the theoretical displacement potential and the numerical results. Very good agreement between the numerical and theoretical distributions is clear, since absolute error is less than $1.2 \cdot 10^{-4}$.

The theoretical radius $r_W \simeq 0.3395$. The contact line is an implicit result of the variational inequality method. We call half-wet elements the elements which have at least one node with zero potential, and at least one node with non-zero potential. We use the center of gravity of these elements and a best fit method [13] to find the parameters of the contact line. Table 1 gives the numerical results for the three meshes used. One can see that the mesh refining method makes the numerical solution tend to the analytical solution. Note that the calculations were performed with three-dimensional code without assumption that the solution is axisymmetric and self-similar.

	dof	r_W
1^{st} mesh	441	0.3329
2^{nd} mesh	1365	0.3330
3^{rd} mesh	2742	0.3361
theory		0.3395

Table 1: Results for cone entry problem

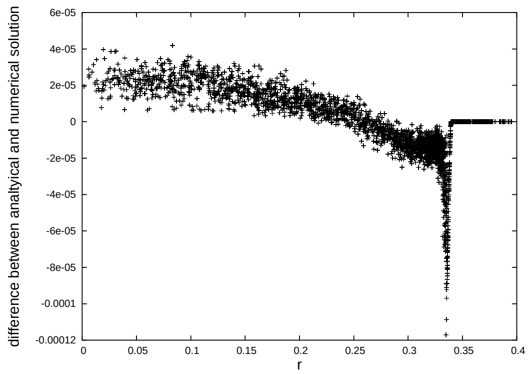

Figure 2: Difference between the numerical solution for the cone entry problem and the analytical solution, on $\{z = 0\}$

An inverse Wagner problem test case

A method to solve the inverse Wagner problem is described in [9]; on the assumption that the contact line is elliptic, for a known contact line with respect to time, for any given velocity or acceleration entry, a method is set up to build the body shape which generates the given contact line. We compute the body shape which generates an elliptic contact line of semi axes $a = \frac{1}{2}\sin\frac{7\pi}{2}t$, and $b = t + 25t^2$, with a zero initial entry velocity, and a constant entering acceleration of 1.
The variational inequality method is supplied to the body computed by the inverse method. The contact line numerically computed should be the contact line that was defined for the inverse Wagner problem.

For a penetration depth $h(t) = \frac{1}{2}t^2 = 0.0006$, we obtain $a \simeq 0.1858$ and $b \simeq 0.0646$. Table 2 presents the results: numerical results tend to the theoretical solution.

	dof	a	b
1^{st} mesh	441	0.1745	0.0561
2^{nd} mesh	1019	0.1789	0.0611
3^{rd} mesh	1406	0.1812	0.0627
4^{th} mesh	2587	0.1838	0.0636
theory		0.1858	0.0646

Table 2: Results for the inverse Wagner problem

The pyramid

To our knowledge, the pyramid impact has no analytical solution. A first asymptotic analysis attempt was performed in [5]. We studied the entry of pyramid $z = 2(|x| + |y|)$, for penetration depth of $h = 0.5$.

Figure 3 shows the displacement potential for this pyramid impact. Figure 4 presents the comparison between the variational inequality method and the first order of the asymptotic method [5]. For the purpose of comparison, the contact line obtained is approximated as a parametric curve:

$$\begin{cases} x(\lambda) = a_0 + \sum_{n=1}^{n_a} a_n \cos(n\lambda) \\ y(\lambda) = b_0 + \sum_{n=1}^{n_b} b_n \sin(n\lambda) \end{cases} \quad (17)$$

and the results are given in table 3 (with $n_a = n_b = 5$).

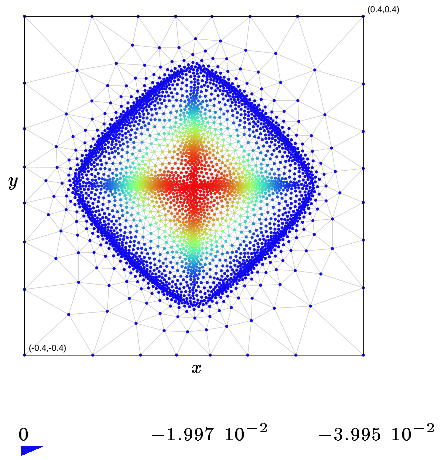

Figure 3: Displacement potential for the pyramid on $\{z = 0\}$

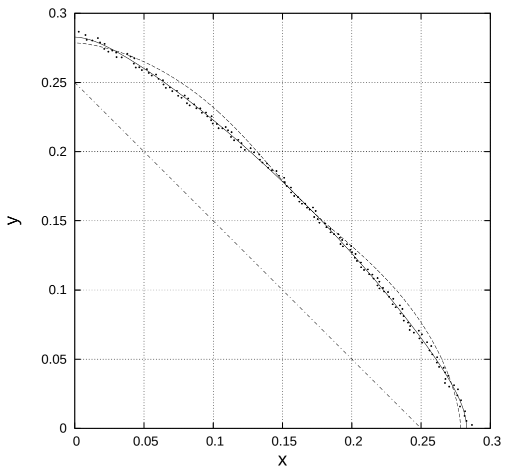

Figure 4: Comparison between the variational inequality method and the asymptotic method [5]. Since the solution is periodic, only a quarter of the domain is plotted. The dot-dashed line is the intersection between the pyramid and the plan $\{z = 0\}$. The points are the center of gravity of the half wet elements. The dashed line is the result of the asymptotic method [5], and the continous line is the result of interpolation given in table 3.

a_0	a_1	a_2	a_3	a_4	a_5
$\simeq 0$	0.2525	0.00108	0.0375	-0.0002	-0.0081
b_0	b_1	b_2	b_3	b_4	b_5
$\simeq 0$	0.2622	-0.0003	-0.0300	-0.0005	-0.0094

Table 3: Results for the pyramid impact

Conclusions

A variational inequality method to solve the three-dimensional Wagner problem is proposed. The well-posedness of the formulation is justified. The problem can be written as a constrained minimization problem.
The comparison between numerical results and analytical solutions shows a good agreement.
A future work consists in precise evaluation of the pressure.

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