Are there trapped modes in the water-wave problem for a freely-floating structure?

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SUMMARY

It has been known for about ten years that, within the framework of the linearised water-wave problem, certain fixed structures can support fluid oscillations of finite energy known as “trapped modes”. In this work the open question of the existence of trapped modes for a freely-floating structure without moorings is considered. For the case of a structure able to move in heave, the conditions necessary for the existence of such a trapped mode are discussed.

1 INTRODUCTION

It is known that, within the linearised theory of water waves, certain structures when held fixed can support a trapped mode of a particular frequency [1]. Such a mode is a free oscillation of the fluid that has finite energy, does not radiate waves to infinity, and in the absence of viscosity will persist for all time. If, for a specified frequency of fluid oscillation, the structure does not support a trapped mode, then the solutions to the frequency-domain radiation and scattering problems at that frequency are unique. From the practical point of view, the existence of a trapped mode means that it is difficult to find numerical solutions to the radiation and scattering problems for a range of frequencies around the trapped-mode frequency [2].

Trapped modes are orthogonal to any incident wave [3] and consequently will not be excited in a scattering problem in either the time or frequency domains. However, the existence of a trapped mode implies that at the trapped-mode frequency there is a pole in a frequency-domain radiation potential [2] and the solution to the corresponding radiation problem does not exist at that frequency [3]. A consequence of this is that trapped modes can be excited in the time domain by the forced oscillations of a trapping structure [4].

Recently, it has been established that the trapped modes supported by a fixed structure, as described above, cannot be excited when that structure is allowed to float freely, with or without incident waves [5, 6]. For motion in a single mode this follows immediately from the equation of motion which shows that the pole in the radiation potential at the trapped-mode frequency is annulled by a corresponding zero in the velocity [5]. However, it is also true that trapped modes supported by fixed structures cannot be excited by motions of the structure in more than one mode [6]. Thus, although the existence of such trapped modes leads to difficulties in the solution of the frequency-domain radiation and scattering problems [2], their existence has no direct relevance to the problem of a freely-floating body in which the radiation and scattering problems are combined through the equation of motion.

The question therefore arises “Are there trapped modes in the water-wave problem for a freely-floating structure?” (such a mode would correspond to a coupled free oscillation of both the structure and the surrounding fluid). The present work seeks to answer this question, although, as yet, it has not been possible to arrive at a definitive answer.

2 FREQUENCY-DOMAIN PROBLEM

For simplicity attention will be restricted to vertical (heave) motions of a structure that may have a single element, or be made up of a number of separate elements that are constrained to move together. It will be assumed initially that the structure is moored and that the moorings have both spring and damper characteristics, although the main interest here is in finding a trapping structure that is not moored. A solution of the governing equations is sought that corresponds to a coupled motion of the fluid and structure about their equilibrium states. The motion should be time harmonic with a particular angular frequency ω and have finite energy.

For time-harmonic motion any time-dependent quantity $\mathcal{F}(t)$ may be written

$$\mathcal{F}(t) = \mathrm{Re}\left\{F\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} \tag{1}$$

where F is, in general, complex. In the absence of any forcing, the equation of motion of the structure in the frequency domain can then be written

$$\left[\rho g W + k - \omega^2(M + a + \mathrm{i}(b+\gamma)/\omega)\right] v = 0. \tag{2}$$

Here ρ is the fluid density, g is the acceleration due to gravity, W is the water plane area, k is the spring constant of the moorings, M is the mass of the structure (which by Archimedes principle is ρ times the submerged volume V), a is the added mass coefficient, b is the damping coefficient, γ is the damping constant of the moorings, and v is the (complex) amplitude of the structural velocity. It follows from (2) that necessary conditions for the existence of a non-zero v (so that the structure is in motion) are that

$$\rho g W + k - \omega^2(M+a) = 0 \tag{3}$$

and

$$b + \gamma = 0. \tag{4}$$

For the fluid motion to have finite energy there can be no radiation of waves to infinity and hence, at the frequency ω, it is required that the damping coefficient b is zero (both a and b are, in general, functions of frequency). It then follows from (4) that the mooring characteristic γ must also be zero. The resonance condition (3) can be satisfied quite easily by an appropriate choice of the spring constant k. Thus, provided a structure can be found such that the damping is zero at the frequency ω, trapped modes for a floating structure moored by springs are readily constructed.

However, the main interest here is in structures without moorings. In which case the construction of a freely-floating trapping structure requires both $b = 0$ and

$$\rho g W - \omega^2(M+a) = 0 \tag{5}$$

at a particular frequency ω.

3 JOHN'S UNIQUENESS PROOF

Over the last fifty years of so considerable attention has been paid to the question of the uniqueness of the solutions to the radiation and scattering problems (uniqueness at a particular frequency is equivalent to the absence of trapped modes). Quite remarkably, given that it is probably the problem of greatest interest, there appears to be only one published result on uniqueness in the problem for a freely-floating structure and that is in the seminal paper by John [7] (although the ideas used in the proof have recently been applied to a structure supported by an air cushion [8]). John's proof is for a structure able to move in all modes of motion, but an outline is given here only for the case of heave motion.

Assume that a trapped mode exists for a freely-floating structure without moorings. John shows that, provided that the structure satisfies the "John condition" (that no line drawn vertically downwards from the free surface intersects the structure) the time-domain velocity potential Φ of the trapped mode satisfies

$$\int_0^{2\pi/\omega} \mathrm{d}t \iint_\Gamma \frac{\partial \Phi}{\partial t}\frac{\partial^2 \Phi}{\partial n \partial t}\,\mathrm{d}S > 0, \tag{6}$$

where Γ is the wetted surface of the structure and the coordinate n is normal to Γ and directed out of the fluid. When interpreted in terms of frequency-domain motion in a single mode this condition is equivalent to $a > 0$ at the trapped mode frequency, and hence it follows from the resonance condition (5) that if a trapped mode exists then

$$\rho g W - \omega^2 M > 0. \tag{7}$$

When ω is sufficiently large this inequality is violated and a contradiction is obtained. Thus, for a structure that satisfies the John condition, $\omega^2 > \rho g W/M$ ensures that there are no trapped modes and hence there is a unique solution to the freely-floating structure problem.

4 WAVE-FREE STRUCTURES

As noted in § 2, a necessary condition for the existence of a trapped mode at a frequency ω is that the damping coefficient must be zero at that frequency. In other words when the structure is forced to oscillate with frequency ω there can be no waves radiated to infinity. A method for the construction of such structures is given by Kyozuka and Yoshida [9].

The idea is as follows. Let ϕ_0 be the potential of a singular solution of the governing equations (excluding the boundary condition on the, as yet, unknown structural surface Γ) that does not radiate waves to infinity at a specified frequency ω. If ϕ_0 is to be the solution to the heave problem then it is required that

$$\frac{\partial \phi_0}{\partial n} = n_z \quad \text{on} \quad \Gamma, \tag{8}$$

where n_z is the component of the inward normal to Γ in the vertical z direction, or equivalently

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{9}$$

where $\phi = z - \phi_0$ and the result $\partial z/\partial n = n_z$ has been used. A suitable surface Γ, such that the correct boundary condition on Γ is satisfied, is determined by examining the streamlines of the flow corresponding to ϕ.

Kyozuka and Yoshida use a combination of a submerged source and a vertical dipole to obtain a wave-free singular solution. To illustrate the idea here a simpler wave-free potential is used and attention is restricted to two dimensions and water of infinite depth. Choose Cartesian coordinates x, z with z measured vertically upwards from the undisturbed free surface, and polar coordinates r, θ that have r measured from the origin and θ measured anticlockwise from the downwards vertical. The potential

$$\phi_0 = \alpha \left\{ \frac{\cos 2\theta}{r^2} + \frac{K \cos\theta}{r} \right\}, \tag{10}$$

where α is a constant and $K = \omega^2/g$, satisfies all of the required conditions and in particular is wave free for all frequencies.

A streamline pattern corresponding to $\phi = z - \phi_0$, where ϕ_0 is given by (10), is shown in figure 1 for particular values of α and K (the pattern is symmetric about $x = 0$). A wave-free structure is obtained from any streamline that isolates the singularity from infinity and that has finite length in $z < 0$. In figure 1 there is only one such streamline; it is the dividing streamline that joins the stagnation point at $x = 0$, $z \approx -0.76$ to the free surface. The structural surface corresponding to this streamline curves towards the origin near the free surface and

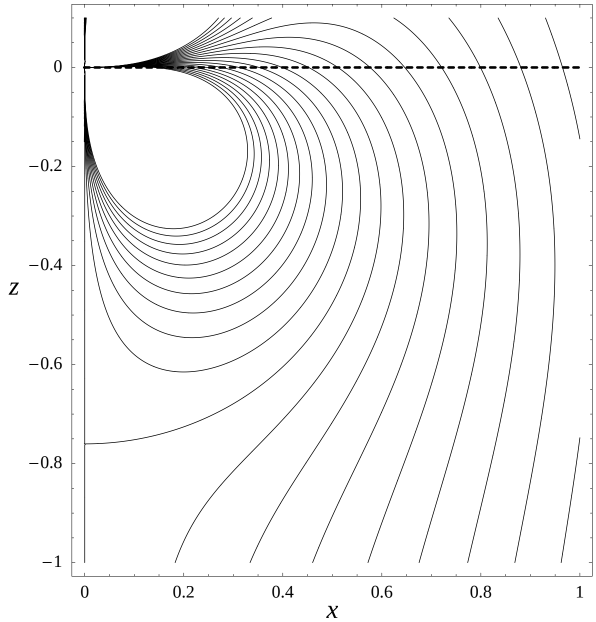

Figure 1: Streamlines corresponding to $\phi = z - \phi_0$; ϕ_0 is given by (10) with $\alpha = 1/2\pi$ and $K = 1$. The dashed line is the free surface.

thus violates the John condition mentioned in § 3 (and hence John's theorem does not apply to this structure).

The streamline pattern varies with α and K. Thus, a structure obtained by this method will be wave free only at the chosen value of K. This is illustrated in figure 2 where the damping coefficient b for the structure corresponding to the dividing streamline in figure 1 (together with its reflection in $x = 0$) is plotted as a function of K. The damping coefficient is zero only at $K = 1$.

5 THE RESONANCE CONDITION

In § 4 it has been shown that it is possible for a structure to oscillate and not to radiate waves so that the damping coefficient is zero at a particular frequency. To obtain a trapped mode it remains to establish whether or not it is possible to satisfy simultaneously the resonance condition (5). Some progress can be made through an application of Green's theorem as follows.

Let S denotes the union of the wetted surface of the structure Γ, the free surface F, and a closing semicircle S_∞ at infinity in $z < 0$. Apply Green's theorem over S to a wave-free potential ϕ_0 and

$$u = z + 1/K \tag{11}$$

which satisfies both Laplace's equation and the linearised free surface condition. It is known that

$$\phi_0 \sim \frac{\mu \cos\theta}{r} \quad \text{as} \quad r \to \infty, \tag{12}$$

where μ is a constant, so that to leading order ϕ_0 is dipole-like at infinity; it is possible that μ may be zero.

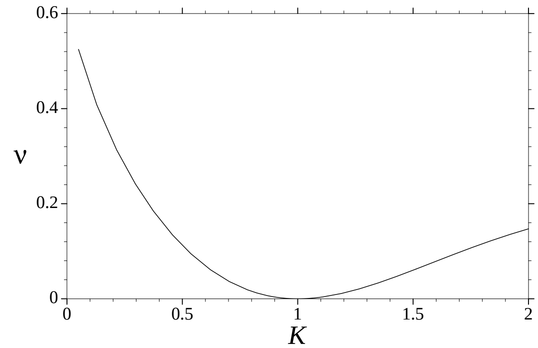

Figure 2: The damping coefficient $\nu = b/(\rho\omega)$ v. the frequency parameter K for a structure formed from the dividing streamline in figure 1.

Green's theorem gives

$$\int_S \left[\phi_0 \frac{\partial u}{\partial n} - u \frac{\partial \phi_0}{\partial n} \right] \mathrm{d}s = 0. \tag{13}$$

The contribution to the integral from F is zero, as both ϕ_0 and u satisfy the free-surface condition, and it follows from the asymptotic form (12) that

$$\int_{S_\infty} \left[\phi_0 \frac{\partial u}{\partial n} - u \frac{\partial \phi_0}{\partial n} \right] \mathrm{d}s = -\pi\mu. \tag{14}$$

From the boundary condition (8),

$$\int_\Gamma \left[\phi_0 \frac{\partial u}{\partial n} - u \frac{\partial \phi_0}{\partial n} \right] \mathrm{d}s = \int_\Gamma \left[\phi_0 n_z - \left(z + \frac{1}{K} \right) n_z \right] \mathrm{d}s. \tag{15}$$

Now, because $b = 0$, the definition of the added mass gives

$$\int_\Gamma \phi_0 n_z \, \mathrm{d}s = \frac{a}{\rho} \tag{16}$$

and, by an application of Stokes' theorem over the submerged volume $V (= M/\rho)$ of the structure,

$$\int_\Gamma z n_z \, \mathrm{d}s = \int_\Gamma z \, \mathrm{d}x = -V. \tag{17}$$

The divergence theorem may be used to show that for an arbitrary ψ

$$\int_{\Gamma + W} \psi n_z \, \mathrm{d}s = -\iint_V \nabla\psi \cdot \mathbf{e}_z \, \mathrm{d}V, \tag{18}$$

where $\mathbf{e}_z$ is a unit vector in the z direction and the normal coordinate n is directed into V. In particular, with $\psi \equiv 1$, this result yields

$$\int_\Gamma n_z \, \mathrm{d}s = -\int_W n_z \, \mathrm{d}s = \int_W \mathrm{d}x = W. \tag{19}$$

Thus, the application of Green's theorem gives

$$\rho g W - \omega^2 (M + a) = -\pi\mu\rho\omega^2 \tag{20}$$

and it is now apparent that the resonance condition (5) cannot be satisfied unless the dipole coefficient μ is zero.

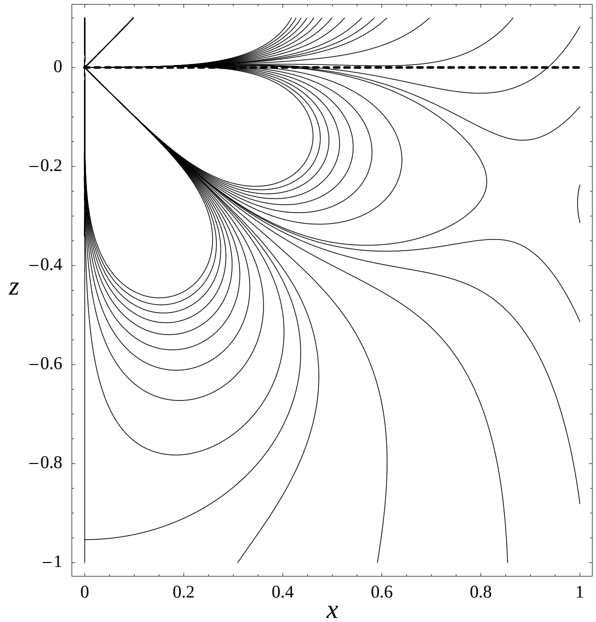

Figure 3: Streamlines corresponding to $\phi = z - \phi_0$; ϕ_0 is given by (21) with $\alpha = 1/2\pi$ and $K = 1$. The dashed line is the free surface.

6 DISCUSSION

It has been shown here that, for a specified frequency of oscillation ω, a trapped mode can exist in the problem of a freely-floating structure without moorings provided that the damping coefficient is zero, and that the resonance condition (5) is satisfied. Both of these requirements are in terms of quantities that are readily calculated from the standard radiation problem in which the structure is forced to perform time-harmonic oscillations.

It is straightforward to find structures that have zero damping at a particular frequency by using wave-free potentials and an example of such a structure is given in § 4. However, the result (20) obtained in § 5 shows that (5) is equivalent to the requirement that the dipole coefficient in the far-field expansion of the radiation potential is zero. In the example of a wave-free structure given here, and in all of those given by Kyozuka and Yoshida [9], the dipole coefficient is not zero.

Singular wave-free potentials with higher-order singularities than a dipole (and hence with $\mu = 0$) are readily obtained. For example, excluding any with a dipole component, the least singular wave-free potential with a singularity in the free surface is

$$\phi_0 = \alpha \left\{ \frac{\cos 4\theta}{r^4} + \frac{K \cos 3\theta}{3r^3} \right\}, \qquad (21)$$

and a streamline pattern for the corresponding ϕ is shown in figure 3. There is still a dividing streamline emanating from a stagnation point, but this now enters the singularity without crossing the free surface and hence does not isolate the singularity from infinity and can not be used to define the surface of a structure. After experimentation with various singularities, and combinations of singularities, it is difficult to see how it is possible to obtain a heaving structure from anything other than a flow that has a dipole-like singularity.

One way to obtain a wave-free structure for which the radiation problem has a flow with a zero dipole coefficient in the far field is by considering two heaving structures that oscillate with opposite phase. However, consideration of the equations of motion shows that the resonance conditions are then modified from those given here for a pure heave oscillation, and again it is difficult to see how they can be satisfied.

Thus, at the time of writing, the author has not been able to establish the existence of trapped modes in the problem of a freely-floating structure. The work is ongoing.

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