Local potential in representation of 3D flow about a ship advancing through regular waves in finite water depth

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1. Introduction

The potential-flow representation for diffraction-radiation by a ship advancing (at speed $\mathcal{U}$) through time-harmonic waves (with frequency ω), in uniform finite water depth, that is given in [1] is considered. This flow representation defines the velocity potential $\widetilde{\phi}$ at a field point $\widetilde{\mathbf{x}}$ in a flow domain as

$$4\pi\widetilde{\phi} = \widetilde{\phi}^{\,R} + i\,\widetilde{\phi}^{\,W} \tag{1a}$$

where $\widetilde{\phi}^{\,R}$ and $\widetilde{\phi}^{\,W}$ represent local-flow and wave components associated with the simple Green function

$$4\pi G = G^R + i\,G^W \tag{1b}$$

given in [2]. The local component $\widetilde{\phi}^{\,R}$ is considered here. This component is defined in [1] as

$$\widetilde{\phi}^{\,R} = \widetilde{\psi}^{\,R} + \widetilde{\chi}^{\,R} \tag{2}$$

with

$$\widetilde{\psi}^{\,R} = \int_{\Sigma_B} d\mathcal{A}\, G^R\,\mathbf{n}\cdot\nabla\phi - \int_{\Sigma_0} dx\,dy\, G^R(\phi_z + F^2\phi_{xx} - f^2\phi + i\widehat{\tau}\,\phi_x) \tag{3a}$$

$$\widetilde{\chi}^{\,R} = \int_{\Sigma_B} d\mathcal{A}\, A_B^R - \int_{\Sigma_D} dx\,dy\, A_D^R + \int_{\Sigma_0} dx\,dy\, a_0^R + \int_{\Gamma} d\mathcal{L}\,(a_\Gamma^R + F^2 G^R t^y\phi_x) \tag{3b}$$

Here, Σ_D represents the sea floor (assumed to be a rigid horizontal wall), Σ_B stands for the mean wetted hull of the ship (or more generally, a geometrical surface that surrounds the ship hull), Σ_0 is the portion of the mean free surface (taken as the plane $z = 0$) located outside the body surface Σ_B, and Γ is the intersection curve between the surfaces Σ_B and Σ_0.

The amplitude functions A_B^R, A_D^R, a_0^R and a_Γ^R in (3b) are defined in [1] as

$$A_B^R = \frac{\mathbf{G}^R}{1+r^3/\ell^3}\cdot\left(\mathbf{n}\times\nabla\phi + \frac{3\,r\,\mathbf{n}\times\mathbf{X}}{\ell^3+r^3}\,\phi\right) - \frac{r^3\phi\,\mathbf{n}\cdot\nabla G^R}{\ell^3+r^3} \tag{4a}$$

$$A_D^R = \frac{r^3 G_z^R\phi}{\ell^3+r^3} + \frac{(G^R)_x^z}{1+r^3/\ell^3}\left(\phi_x + \frac{3\,r\,X\phi}{\ell^3+r^3}\right) + \frac{(G^R)_y^z}{1+r^3/\ell^3}\left(\phi_y + \frac{3\,r\,Y\phi}{\ell^3+r^3}\right) \tag{4b}$$

$$a_0^R = \frac{r^3\pi^R\phi}{\ell^3+r^3} + \frac{(\pi^R)_x^{zz}}{1+r^3/\ell^3}\left(\phi_x + \frac{3\,r\,X\phi}{\ell^3+r^3}\right) + \frac{(\pi^R)_y^{zz}}{1+r^3/\ell^3}\left(\phi_y + \frac{3\,r\,Y\phi}{\ell^3+r^3}\right) \tag{4c}$$

$$a_\Gamma^R = f^2\phi\,\frac{t^y(G^R)_x^{zz} - t^x(G^R)_y^{zz}}{1+r^3/\ell^3} - (F^2 G_x^R - i\widehat{\tau}G^R)\frac{r^3 t^y\phi}{\ell^3+r^3}$$
$$- \frac{(F^2 G_x^R - i\widehat{\tau}G^R)_y^{zz}}{1+r^3/\ell^3}\left(\mathbf{t}\cdot\nabla\phi + \frac{3\,r\,\mathbf{t}\cdot\mathbf{X}}{\ell^3+r^3}\,\phi\right) \tag{4d}$$

$\mathbf{n} = (n^x, n^y, n^z)$ in (4a) stands for a unit vector that is normal to the body surface Σ_B and points into the flow domain; $\mathbf{t} = (t^x, t^y, 0)$ in (4d) is a unit vector tangent to the boundary curve Γ, which is oriented clockwise (looking down); $f = \omega\sqrt{L/g}$ is the nondimensional wave frequency; $F = \mathcal{U}/\sqrt{gL}$ is the Froude number; and $\widehat{\tau} = 2fF$. Furthermore, $\mathbf{G}^R$ in (4a) and π^R in (4c) are defined as

$$\mathbf{G}^R = [\,(G^R)_y^z, -(G^R)_x^z, 0\,] \qquad\qquad \pi^R = G_z^R + F^2 G_{xx}^R - f^2 G^R - i\widehat{\tau}G_x^R \tag{5}$$

and a subscript/superscript attached to $\mathbf{G}^R$ or π^R means differentiation/integration, respectively. In (4), r represents the distance between a point $\widetilde{\mathbf{x}}$ in the flow domain and a point $\mathbf{x}$ at the boundary surface $\Sigma_D\cup\Sigma_0\cup\Sigma_B$, i.e.

$$r = \sqrt{\mathbf{X}\cdot\mathbf{X}} \qquad \text{with} \qquad \mathbf{X} = (X, Y, Z) = \widetilde{\mathbf{x}} - \mathbf{x} = (\widetilde{x} - x, \widetilde{y} - y, \widetilde{z} - z) \tag{6}$$

Finally, ℓ is a positive real number that controls the transition between the classical and weakly-singular potential representations that are given in [3] and are contained in the generalized representation (1a), given in [1] and considered here.

2. Local components in Green function and related functions

The local component G^R in (1b) is chosen here as that given in [2]. This local component is defined by four elementary free-space Rankine sources that account for the dominant terms in both the nearfield and farfield asymptotic approximations to the non-oscillatory local-flow component contained in the Green function associated with wave diffraction-radiation with forward speed (and the special cases corresponding to $\mathcal{U}=0$ or $\omega=0$). Specifically, the local Rankine component G^R is chosen here as

$$G^R = -1/r + 1/r_* - 2/r_F + 2/r_{Ff} \tag{7}$$

where r is given by (6), and $r_*\,, r_F\,, r_{Ff}$ are defined as

$$\begin{array}{lll} r_* = \sqrt{\mathbf{X}_*\cdot\mathbf{X}_*} & \mathbf{X}_* = (X, Y, Z_*) & Z_* = \widetilde{z} + z \\ r_F = \sqrt{\mathbf{X}_F\cdot\mathbf{X}_F} & \mathbf{X}_F = (X, Y, Z_F) & Z_F = Z_* - F^2 \\ r_{Ff} = \sqrt{\mathbf{X}_{Ff}\cdot\mathbf{X}_{Ff}} & \mathbf{X}_{Ff} = (X, Y, Z_{Ff}) & Z_{Ff} = Z_F - 1/f^2 \end{array} \tag{8a}$$

with X and Y given by (6). Define

$$r_C = \sqrt{\mathbf{X}_C\cdot\mathbf{X}_C} \qquad \mathbf{X}_C = (X, Y, Z_C) \qquad Z_C = Z_* - C^2 \tag{8b}$$

Thus, $r_*\,, r_F\,, r_{Ff}$ correspond to C^2 equal to $0\,, F^2, F^2+1/f^2$, respectively.

Expressions (6) yield

$$\left\{\begin{array}{c}(1/r)^{zz}_x \\ (1/r)^{zz}_y\end{array}\right\} = \frac{1}{r+|Z|}\left\{\begin{array}{c}X \\ Y\end{array}\right\} \qquad (1/r)^{zz}_{xy} = \frac{XY/r}{(r+|Z|)^2} \tag{9a}$$

$$(1/r)^z = -\text{sign}(Z)\,\ln(r+|Z|) \qquad \left\{\begin{array}{c}(1/r)^z_x \\ (1/r)^z_y\end{array}\right\} = \frac{\text{sign}(Z)}{r+|Z|}\,\frac{1}{r}\left\{\begin{array}{c}X \\ Y\end{array}\right\} \tag{9b}$$

Similarly, (8) yield

$$\left\{\begin{array}{c}(1/r_C)^{zz}_x \\ (1/r_C)^{zz}_y\end{array}\right\} = \frac{1}{r_C+|Z_C|}\left\{\begin{array}{c}X \\ Y\end{array}\right\} \qquad (1/r_C)^{zz}_{xy} = \frac{XY/r_C}{(r_C+|Z_C|)^2} \tag{10a}$$

$$(1/r_C)^z = \text{sign}(Z_C)\,\ln(r_C+|Z_C|) \qquad \left\{\begin{array}{c}(1/r_C)^z_x \\ (1/r_C)^z_y\end{array}\right\} = \frac{-\,\text{sign}(Z_C)}{r_C+|Z_C|}\,\frac{1}{r_C}\left\{\begin{array}{c}X \\ Y\end{array}\right\} \tag{10b}$$

Expressions (7), (9), (10) and (5) yield

$$\left\{\begin{array}{c}(G^R)^z_x \\ (G^R)^z_y\end{array}\right\} = \widetilde{B}_2\left\{\begin{array}{c}X \\ Y\end{array}\right\} \qquad \mathbf{G}^R = \widetilde{B}_2\,(Y, -X, 0) \tag{11}$$

where $\widetilde{B}_2$ is defined below by (17) with (19d) and (19e). Expressions (11) yield

$$\mathbf{G}^R\cdot(\mathbf{n}\times\nabla\phi) = \widetilde{B}_2\,[\,Y(\mathbf{n}\times\nabla\phi)^x - X(\mathbf{n}\times\nabla\phi)^y\,] \tag{12a}$$

$$\mathbf{G}^R\cdot(\mathbf{n}\times\mathbf{X}) = \widetilde{B}_2\,[\,(n^xX + n^yY)\,Z - n^z(X^2+Y^2)\,] \tag{12b}$$

Expressions (7), (6) and (8) yield

$$G^R_z = -B_2 \qquad \mathbf{n}\cdot\nabla G^R = -(n^xX + n^yY)B_3 - n^zB_2 \tag{12c}$$

where B_2 and B_3 are defined by (17) with (19b), (19d) and (19e).

At the free-surface plane $z=0$, expressions (6)–(10) yield

$$\begin{array}{ll} G^R = -2R_1 \qquad G^R_x = -2XR_3 \qquad \left\{\begin{array}{c}(G^R)^{zz}_x \\ (G^R)^{zz}_y\end{array}\right\} = -2\widetilde{R}_1\left\{\begin{array}{c}X \\ Y\end{array}\right\} & (G^R)^{zz}_{xy} = -2XY\widetilde{R}_3 \\ G^R_{xx} = 2(R_3 - 3X^2R_5) & (G^R)^{zz}_{xx} = 2(\widetilde{R}_1 - X^2\widetilde{R}_3) \\ (G^R)^{zz}_{xxx} = 2(3\widetilde{R}_3 - X^2\widetilde{R}_5)X & (G^R)^{zz}_{xxy} = 2(\widetilde{R}_3 - X^2\widetilde{R}_5)Y \\ G^R_z = 2(R^*_2 - R_2) & \left\{\begin{array}{c}(G^R)^z_x \\ (G^R)^z_y\end{array}\right\} = 2(\widetilde{R}^*_2 - \widetilde{R}_2)\left\{\begin{array}{c}X \\ Y\end{array}\right\} \end{array} \tag{13}$$

where $R_1\,,\widetilde{R}_1\,,R_3\,,\widetilde{R}_3\,,R_5\,,\widetilde{R}_5\,,R_2\,,\widetilde{R}_2\,,R_2^*\,,\widetilde{R}_2^*$ are defined by (19). Finally, (5) and (13) yield

$$\pi^R = 2P_1 \qquad (\pi^R)_y^{zz} = 2Y\widetilde{P}_1 \qquad (\pi^R)_x^{zz} = 2(X\widetilde{P}_1 + \widetilde{P}_2) \tag{14}$$

with $P_1\,,\widetilde{P}_1$ and $\widetilde{P}_2$ given by (18).

3. Local component in generalized potential-flow representation

Expressions (3) for the potentials $\widetilde{\psi}^R$ and $\widetilde{\chi}^R$, at a flow-field point $\widetilde{\mathbf{x}}$, in the representation (2) of the local-flow potential $\widetilde{\phi}^R$ in the local/wave decomposition (1a) yield

$$\widetilde{\psi}^R = \int_{\Sigma_B} d\mathcal{A}\left(\frac{-1}{r} + \frac{1}{r_*} - 2R_1\right)\mathbf{n}\cdot\nabla\phi + 2\int_{\Sigma_0} dx\,dy\,R_1(\phi_z + F^2\phi_{xx} - f^2\phi + i\widehat{\tau}\,\phi_x) \tag{15a}$$

$$\widetilde{\chi}^R = \int_{\Sigma_B} d\mathcal{A}\,A_B^R - \int_{\Sigma_D} dx\,dy\,A_D^R + 2\int_{\Sigma_0} dx\,dy\,A_0^R + 2\int_\Gamma d\mathcal{L}\,(A_\Gamma^R - F^2R_1t^y\phi_x) \tag{15b}$$

One has $R_1 = 1/r_F - 1/r_{Ff}$ in (15a), as given by (19a), and $A_0^R = a_0^R/2$ and $A_\Gamma^R = a_\Gamma^R/2$ in (15b). Expressions (4) and (11)–(14) show that the functions $A_B^R\,,A_D^R\,,A_0^R$ and A_Γ^R in (15b) are given by

$$A_B^R = \left[(Xn^x + Yn^y)\left(\frac{3\,Z\,r\,\widetilde{B}_2}{1+r^3/\ell^3} + r^3B_3\right) - n^z\left(\frac{3(X^2+Y^2)\,r\,\widetilde{B}_2}{1+r^3/\ell^3} - r^3B_2\right)\right]\frac{\phi}{\ell^3+r^3}$$
$$+\widetilde{B}_2\,\frac{Y(\mathbf{n}\times\nabla\phi)^x - X(\mathbf{n}\times\nabla\phi)^y}{1+r^3/\ell^3} \tag{16a}$$

$$A_D^R = \left(\frac{3(X^2+Y^2)\,r\,\widetilde{B}_2}{1+r^3/\ell^3} - r^3B_2\right)\frac{\phi}{\ell^3+r^3} + \widetilde{B}_2\,\frac{X\phi_x + Y\phi_y}{1+r^3/\ell^3} \tag{16b}$$

$$A_0^R = \frac{r^3P_1\,\phi}{\ell^3+r^3} + \frac{X\widetilde{P}_1 + \widetilde{P}_2}{1+r^3/\ell^3}\left(\phi_x + \frac{3\,r\,X\phi}{\ell^3+r^3}\right) + \frac{Y\widetilde{P}_1}{1+r^3/\ell^3}\left(\phi_y + \frac{3\,r\,Y\phi}{\ell^3+r^3}\right) \tag{16c}$$

$$A_\Gamma^R = f^2\widetilde{R}_1\,\frac{Yt^x - Xt^y}{1+r^3/\ell^3}\,\phi + \frac{F^2XR_3 - i\widehat{\tau}R_1}{\ell^3+r^3}\,r^3\,t^y\,\phi$$
$$+\frac{F^2X\widetilde{R}_3 - i\widehat{\tau}\widetilde{R}_1}{1+r^3/\ell^3}\,Y\left(\mathbf{t}\cdot\nabla\phi + 3\,r\,\frac{Xt^x + Yt^y}{\ell^3+r^3}\,\phi\right) \tag{16d}$$

X, Y, Z and $r\,,r_*\,,r_F\,,r_{Ff}$ are given by (6) and (8a). The functions $\widetilde{B}_2\,,B_2$ and B_3 in (16a) and (16b) are defined as

$$\widetilde{B}_2 = \frac{1}{r}\,\frac{\text{sign}(-Z)}{r+|Z|} + \widetilde{R}_2^* - 2\widetilde{R}_2 \qquad B_2 = \frac{Z}{r^3} - R_2^* + 2R_2 \qquad B_3 = \frac{1}{r^3} - \frac{1}{r_*^3} + 2R_3 \tag{17}$$

with (19e), (19d) and (19b). The functions $P_1\,,\widetilde{P}_1\,,\widetilde{P}_2$ in (16c) are defined as

$$\begin{aligned} P_1 &= f^2R_1 + F^2(R_3 - 3X^2R_5) + i\widehat{\tau}\,XR_3 + R_2^* - R_2 \\ \widetilde{P}_1 &= f^2\widetilde{R}_1 + F^2(\widetilde{R}_3 - X^2\widetilde{R}_5) + i\widehat{\tau}\,X\widetilde{R}_3 + \widetilde{R}_2^* - \widetilde{R}_2 \\ \widetilde{P}_2 &= 2F^2X\widetilde{R}_3 - i\widehat{\tau}\,\widetilde{R}_1 \end{aligned} \tag{18}$$

The functions $R_1\,,\widetilde{R}_1\,,R_3\,,\widetilde{R}_3\,,R_5\,,\widetilde{R}_5\,,R_2\,,\widetilde{R}_2\,,R_2^*\,,\widetilde{R}_2^*$ in (16d), (17) and (18) are given by

$$R_1 = \frac{1}{r_F} - \frac{1}{r_{Ff}} \qquad \widetilde{R}_1 = \frac{1}{r_F + |Z_F|} - \frac{1}{r_{Ff} + |Z_{Ff}|} \tag{19a}$$

$$R_3 = \frac{1}{r_F^3} - \frac{1}{r_{Ff}^3} \qquad \widetilde{R}_3 = \frac{1}{r_F}\,\frac{1}{(r_F + |Z_F|)^2} - \frac{1}{r_{Ff}}\,\frac{1}{(r_{Ff} + |Z_{Ff}|)^2} \tag{19b}$$

$$R_5 = \frac{1}{r_F^5} - \frac{1}{r_{Ff}^5} \qquad \widetilde{R}_5 = \frac{1}{r_F^3}\,\frac{3\,r_F + |Z_F|}{(r_F + |Z_F|)^3} - \frac{1}{r_{Ff}^3}\,\frac{3\,r_{Ff} + |Z_{Ff}|}{(r_{Ff} + |Z_{Ff}|)^3} \tag{19c}$$

$$R_2 = \frac{-Z_F}{r_F^3} - \frac{-Z_{Ff}}{r_{Ff}^3} \qquad \widetilde{R}_2 = \frac{1}{r_F}\,\frac{\text{sign}(-Z_F)}{r_F + |Z_F|} - \frac{1}{r_{Ff}}\,\frac{\text{sign}(-Z_{Ff})}{r_{Ff} + |Z_{Ff}|} \tag{19d}$$

$$R_2^* = \frac{-Z_*}{r_*^3} \qquad \widetilde{R}_2^* = \frac{1}{r_*}\,\frac{\text{sign}(-Z_*)}{r_* + |Z_*|} \tag{19e}$$

4. Conclusion

The generalized flow representation for diffraction-radiation by a ship advancing through time-harmonic waves, in uniform finite water depth, given in [1] defines the velocity potential $\widetilde{\phi}$ at a field point $\widetilde{\mathbf{x}}$ in the flow domain as the sum of a local component $\widetilde{\phi}^R$ and a wave component $\widetilde{\phi}^W$, in accordance with (1a), that are associated with the corresponding decomposition (1b) of the simple Green function given in [2]. The local potential $\widetilde{\phi}^R$ in the local/wave decomposition (1a) is considered here.

Expression (2) defines the local potential $\widetilde{\phi}^R$ as the sum of two components $\widetilde{\psi}^R$ and $\widetilde{\chi}^R$. These components are defined by (15) in terms of distributions of elementary free-space Rankine sources, and of the related simple algebraic functions (17)–(19), over the mean wetted ship hull Σ_B (or a geometrical surface that surrounds the ship hull), the portion Σ_0 of the mean free surface located outside the body surface Σ_B, the intersection curve Γ between the surfaces Σ_B and Σ_0, and the horizontal sea floor Σ_D.

The component $\widetilde{\psi}^R$ defined by (15a) involves a distribution of elementary Rankine sources over the body surface Σ_B, with strength equal to the normal component $\mathbf{n}\cdot\nabla\phi$ of the velocity $\nabla\phi$, and the free surface Σ_0. The integral over the free surface Σ_0 in (15a) is null if the potential ϕ is assumed to satisfy the usual linearized free-surface boundary condition. However, this free-surface integral is not null for a surface-effect ship; or for linearization about a base flow (e.g. double-body flow) that differs from the uniform stream opposing the ship speed, as is allowed in the potential representation given in [1] and used here. In any case, the free-surface pressure $\phi_z + F^2\phi_{xx} - f^2\phi + i\widehat{\tau}\,\phi_x$ in (15a) is null outside a compact nearfield region of the free surface Σ_0 bordering the boundary curve Γ.

The component $\widetilde{\chi}^R$ defined by (15b), (16) and (17)–(19) involves the potential ϕ and the tangential velocity components $\mathbf{n}\times\nabla\phi$ (at the body surface Σ_B), ϕ_x and ϕ_y (at the sea floor Σ_D and the free surface Σ_0), and $\mathbf{t}\cdot\nabla\phi$ (at the boundary curve Γ). The line integral around Γ in (15b) also involves the velocity component ϕ_x, which can be expressed in terms of the tangential velocity component $\mathbf{t}\cdot\nabla\phi$ and the velocity component $\mathbf{n}^\Gamma\cdot\nabla\phi$ along a unit vector $\mathbf{n}^\Gamma = (-t^y, t^x, 0)$ normal to the curve Γ in the free-surface plane [3]. The amplitude functions A_D^R and A_0^R in the integrals over the sea floor Σ_D and the free surface Σ_0 in (15b) vanish rapidly in the horizontal farfield $h = \sqrt{x^2+y^2} \to \infty$. Specifically, (16b), (16c) and (17)–(19) show that A_D^R and A_0^R are $O(\phi/h^3)$ as $h\to\infty$. Thus, the integrals over the sea floor Σ_D and the free surface Σ_0 in (15b) in fact only involve numerical integration over compact nearfield regions of the unbounded horizontal planes Σ_D and Σ_0. In the nearfield limit $r\to 0$, the integrand of the line integral around the curve Γ in (15b) is finite, and the integrands A_B^R, A_D^R, A_0^R of the three surface integrals in (15b) are no more singular than the elementary Rankine sources in (15a), i.e. are weakly singular. The potential $\widetilde{\chi}^R$ defined by the representation (15b) accordingly is continuous at the boundary surface $\Sigma_D\cup\Sigma_0\cup\Sigma_B$ (whereas the potential defined by the classical potential-flow representation, which involves a dipole distribution, is discontinuous at the boundary surface).

Thus, the representation (15) of the local potentials $\widetilde{\psi}^R$ and $\widetilde{\chi}^R$ only requires integration over compact nearfield surfaces. Furthermore, the integrands of the integrals in (15) are weakly singular in the nearfield, and only involve ordinary algebraic functions (related to four elementary free-space Rankine sources). This representation of the local potential $\widetilde{\phi}^R$ in the local/wave decomposition (1a) therefore is particularly simple and provides a practical basis for numerical evaluation. The representation is valid also in the special cases $F=0$ or $f=0$. The wave potential $\widetilde{\phi}^W$ in (1a) is defined in [1] by single (one-fold) Fourier integrals that involve spectrum functions given by boundary distributions of elementary waves. This Fourier-Kochin representation of the wave potential, based on the generalized potential representation given in [1] and the simple Green function given in [2], also yields a practical basis for numerical evaluation. The complementary representation of the wave potential is given elsewhere.

References

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