The 20th International Workshop on Water Waves and Floating Bodies
Longyearbyen, Spitsbergen (Norway), 29th May - 1st June 2005

On the Dirichlet-Neumann Operator for Nonlinear Water Waves

HEMMING A. SCHÄFFER
SchäfferWaves, Sortedam Dossering 59 H st., DK-2100 Copenhagen Ø, Denmark
(Hemming@SchafferWaves.dk)

A series expansion of the Dirichlet-Neumann operator was derived by Craig & Sulem (1993) and in a slightly different form by Bateman, Swan & Taylor (2001). These are supposedly superior to techniques derived earlier by West et al. (1987) and Dommermuth & Yue (1987) under seemingly more restrictive assumptions. This paper extracts the Dirichlet-Neumann operator expansions from West et al. and from Dommermuth & Yue. With regard to the operator expansions alone it is found that Bateman et al. is identical to Dommermuth & Yue and to West et al., while Craig and Sulem is slightly different. However, wave transformation is not determined by this expansion alone. The way the expansion is used in the free surface boundary conditions also matters. As a consequence, Craig & Sulem is found to be identical with West et al. while Bateman et al. is identical to Dommermuth & Yue, when it comes to the determination of the temporal derivative of the surface elevation. This is due to the consistent truncation at nonlinear order devised by West et al.

1. Introduction

The Dirichlet-Neumann (DN) operator for water waves expresses essentially the normal surface particle velocity in terms of the velocity potential at the surface. Given a procedure for estimating the DN operator (subject to lateral boundary conditions), the water-wave problem is reduced to the simultaneous time-integration of the kinematic and the dynamic free surface boundary conditions (KFSBC and DFSBC, respectively). Thus, the dependent variables are evaluated at the free surface only, also if the water depth is finite, and even if it is variable.

A Taylor series expansion of the DN operator was derived by Craig and Sulem (CS, 1993) and in a slightly different form by Bateman, Swan and Taylor (BST, 2001). Without using the terminology of a DN operator, West et al. (WW, 1987) and Dommermuth and Yue (DY, 1987) had already derived related expressions under apparently more restrictive assumptions. Here West et al. (1987) is referred to as WW since this makes a nice superscript for symbols used below and at the same time it acknowledges Watson and West (1975), who West et al. quote for their methodology.

Although the DN operator expansions of CS and BST seem quite different from the work of WW and DY, this paper shows that they are not so different after all.

Common to all the methods described in this paper is that they involve high-order derivatives and nonlinearities by which they fail to converge for extremely steep waves. A fast method overcoming this problem was developed by Clamond and Grue (2001). Their method is based on boundary integrals and for variable depth this is at the expense of having to solve for additional dependent variables defined along the bottom.

The present analysis was carried out in search for the preferred method for including nonlinearity in an ongoing development of a new wave model. This model is based on convolution integrals in physical space and involves only variables at the surface although the bottom may be variable with mild slopes (Schäffer 2003, 2004).

Throughout this paper expressions are given for one horizontal dimension, but extension to two horizontal dimensions is straightforward.

2. Governing equations

Including a small ordering parameter, ε, the KFSBC and DFSBC expressed in terms of the surface elevation η and the velocity potential at the surface $\tilde{\Phi} = \Phi(z = \eta)$ read

$$\eta_t = (1 + \varepsilon^2 \eta_x^2)\tilde{w} - \varepsilon \eta_x \tilde{\Phi}_x \tag{1}$$

$$\tilde{\Phi}_t = -g\eta - \tfrac{1}{2}\varepsilon\left(\tilde{\Phi}_x^2 - \tilde{w}^2(1 + \varepsilon^2 \eta_x^2)\right) \tag{2}$$

as used by WW, DY and essentially also by BST. To march η and $\tilde{\Phi}$ forward in time a closure is needed. An attractive explicit form is

$$\tilde{w} = G^{\mathrm{WW}}(\tilde{\Phi}) \tag{3}$$

where G is essentially a DN operator dependent on η and operating on $\tilde{\Phi}$. G further depends on the constant or variable water depth, h, and it may also account for non-periodic lateral boundary conditions.

A slightly different definition of the DN operator was introduced by CS as

$$\eta_t = G^{CS}(\tilde{\Phi}) \tag{4}$$

This directly serves as the KFSBC and the appropriate form of the DFSBC condition becomes

$$\tilde{\Phi}_t = -g\eta - \frac{\varepsilon}{2(1 + \varepsilon^2 \eta_x^2)}(\tilde{\Phi}_x^2 - \eta_t^2 - 2\varepsilon \eta_x \tilde{\Phi}_x \eta_t) \tag{5}$$

3. The Dirichlet-Neumann operator

In this section, WW's procedure for accounting for wave nonlinearity is detailed, and the equivalent expansion of the DN operator is explicitly extracted. Following DY yields identical results for the DN operator. However, as discussed subsequently, WW and DY differ in how the operator is used in the surface boundary conditions.

The surface potential is expressed as the perturbation series

$$\varepsilon\tilde{\Phi} = \sum_{m=1}^{M} \varepsilon^m \tilde{\Phi}^{(m)} \tag{6}$$

With a vertical axis, z, originating at still water level, each term is expressed as a Taylor series

$$\tilde{\Phi}^{(m)} = \sum_{n=0}^{M-m} \varepsilon^n \frac{\eta^n}{n!}\frac{\partial^n \Phi_0^{(m)}}{\partial z^n} \tag{7}$$

by which

$$\varepsilon\tilde{\Phi} = \sum_{m=1}^{M}\sum_{n=0}^{M-m} \varepsilon^{m+n} \frac{\eta^n}{n!}\frac{\partial^n \Phi_0^{(m)}}{\partial z^n} \tag{8}$$

Collecting terms of order ε, this may be rewritten as

$$\varepsilon\tilde{\Phi} = \sum_{m=1}^{M}\sum_{n=0}^{m-1} \varepsilon^{m} \frac{\eta^n}{n!}\frac{\partial^n \Phi_0^{(m-n)}}{\partial z^n} \tag{9}$$

Requiring this to be valid for all values of ε gives

$$\Phi_0^{(1)} = \tilde{\Phi} \tag{10}$$

at order ε, while the ε^m-terms result in

$$\Phi_0^{(m)} = -\sum_{n=1}^{m-1} \frac{\eta^n}{n!}\frac{\partial^n \Phi_0^{(m-n)}}{\partial z^n} \tag{11}$$

This conveniently expresses $\Phi_0^{(m)}$ in terms of lower order expressions already known from the previous steps of this recursion. At each order the z-derivatives are computed from the recursion relation

$$\frac{\partial^n \Phi_0^{(m)}}{\partial z^n} = D^2 \frac{\partial^{n-2}\Phi_0^{(m)}}{\partial z^{n-2}} \tag{12}$$

where $D = -i\partial_x$, which holds due to $\Phi_0^{(m)}$ satisfying the Laplace equation. The two starting values (indices $n = 0$ and $n = 1$) needed for this recursion are $\Phi_0^{(m)}$ and

$$\frac{\partial \Phi_0^{(m)}}{\partial z} = G_0(\Phi_0^{(m)}) \tag{13}$$

Here

$$G_0 = D \tanh hD \tag{14}$$

is a Fourier multiplier representing the 0th order DN operator, that follows directly from small amplitude wave theory. Variable depth and/or non-periodic lateral boundary conditions require that G_0 be adjusted accordingly.

The derivations (6)-(11) remain valid if the potential is replaced by the vertical velocity, w, and with

$$\varepsilon\, \tilde{w} = \sum_{m=1}^{M} \varepsilon^m \tilde{w}^{(m)} \tag{15}$$

we get

$$\tilde{w}^{(m)} = \sum_{n=0}^{m-1} \frac{\eta^n}{n!} \frac{\partial^{n+1}\Phi_0^{(m-n)}}{\partial z^{n+1}} \tag{16}$$

The DN-operator formulation (3) is expanded as

$$\varepsilon\, \tilde{w} = \sum_{m=1}^{M} \varepsilon^m G_{m-1}^{\mathrm{WW}}(\tilde{\Phi}) \tag{17}$$

The recipe for evaluating $\tilde{w}^{(m)}$ at any order and extracting the associated terms of the DN operator, G_m^{WW}, is now as follows: Evaluate (16) obtaining the z-derivatives from the recursion relation (12) with starting values (11) & (13), where (11) is a recursion relation initialized with (10). The first three terms are given by

$$G_0^{\mathrm{WW}} = G_0 \tag{18}$$

$$G_1^{\mathrm{WW}} = \eta D^2 - G_0 \eta G_0 \tag{19}$$

$$G_2^{\mathrm{WW}} = G_0\eta G_0 \eta G_0 - \frac{1}{2} G_0 \eta^2 D^2 - \eta D^2 \eta G_0 + \frac{1}{2}\eta^2 D^2 G_0 \tag{20}$$

This exactly matches the DN operator derived by BST (their 22a-c) as a slight variation of CS. For the special case of deep water, where $G_0 = D$, (18)-(20) in (17) reduces to the expression given already in 1975 by Watson and West (eq. A5 in their Appendix).

To determine the slightly different DN operator of CS, (4) and (3) are substituted into the KFSBC (1), to get

$$G^{\mathrm{CS}}(\tilde{\Phi}) = (1 + \varepsilon^2 \eta_x^2) G^{\mathrm{WW}}(\tilde{\Phi}) - \varepsilon \eta_x \tilde{\Phi}_x \tag{21}$$

Expanding the DN operators and collecting the orders gives

$$G_0^{\mathrm{CS}} = G_0 \tag{22}$$

$$G_1^{\mathrm{CS}} = D\eta D - G_0 \eta G_0 \tag{23}$$

$$G_2^{\mathrm{CS}} = G_0 \eta G_0 \eta G_0 - \frac{1}{2} G_0 \eta^2 D^2 - \frac{1}{2} D^2 \eta^2 G_0 \tag{24}$$

where the explicit appearance of the surface elevation gradient has been eliminated by incorporating these in the operators (The last term in G_2^{CS} accommodates the two last terms in G_2^{WW} as well as the term arising from G_0^{WW}).

These results are identical to those of CS (their eq. 2.14). This contradicts the beliefs of CS, who describe their DN operator expansion as "uniformly valid in wavenumber", while they claim that it "differs from the spectral methods of both West et al. and Dommermuth and Yue, where both Φ and η are assumed to be $O(\varepsilon)$ quantities, and the expansion is not uniform in wavenumber".

According to (1) or (4), errors in the DN operator affect the determination of η_t already at a linear level. On the other hand $\tilde{\Phi}_t$ is less sensitive to errors, since the DN operator only appears in the nonlinear terms of the DFSBC, (2) or (5). Consequently, we skip the analysis of $\tilde{\Phi}_t$ and concentrate on η_t.

WW devised a consistent truncation with regard to nonlinear order in connection with the evaluation of η_t (and $\tilde{\Phi}_t$). It turns out that their method corresponds exactly to the procedure outlined above for evaluating G_m^{CS} from G_m^{WW}. This implies that $\eta_t^{\mathrm{CS}} = \eta_t^{\mathrm{WW}} \neq \eta_t^{\mathrm{BST}} = \eta_t^{\mathrm{DY}}$, although $G_m^{\mathrm{CS}} \neq G_m^{\mathrm{WW}} = G_m^{\mathrm{BST}} = G_m^{\mathrm{DY}}$ as summarized in Table 1. These results were backed up by a numerical example omitted here.

As a final remark, a DN operator is strictly one that maps $\tilde{\Phi}$ to the normal surface particle velocity, $\eta_t / \sqrt{1+\varepsilon^2\eta_x^2}$. However, with a consistent truncation involving a Taylor expansion of $1/\sqrt{1+\varepsilon^2\eta_x^2}$, this choice turns out to produce the same $\eta_t = \eta_t^{\mathrm{CS}} = \eta_t^{\mathrm{WW}}$.

Reference		DN operator	η_t
West et al. (1987); (Watson & West, 1975)	WW	G^{WW}	η_t^{WW}
Dommermuth & Yue (1987)	DY	$G^{\mathrm{DY}} = G^{\mathrm{WW}}$	η_t^{DY}
Craig & Sulem (1993)	CS	G^{CS}	$\eta_t^{\mathrm{CS}} = \eta_t^{\mathrm{WW}}$
Bateman, Swan & Taylor (2001)	BST	$G^{\mathrm{BST}} = G^{\mathrm{WW}}$	$\eta_t^{\mathrm{BST}} = \eta_t^{\mathrm{DY}}$

Table 1. The Dirichlet-Neumann (DN) operator as given by (or extracted from) different authors. Note that the designation on η_t does not always match that on the DN operator, *G*.

References

Bateman, W.J.D, Swan, C. and Taylor, P.H.(2001). On the efficient numerical simulation of directionally spread surface water waves. *J. Comp. Phys*. **174**, 277-305.

Clamond, D. and Grue, J. (2001). A fast method for fully nonlinear water-wave computations. *J. Fluid Mech*. **447**, 337-355.

Craig, W. and Sulem, C. (1993). Numerical simulation of gravity waves. *J. Comp. Phys*. **108**, 73-83.

Dommermuth, D.G. and Yue, D.K.Y. (1987). A high-order spectral method for the study of nonlinear gravity waves. *J. Fluid Mech.* **184**, 267-288.

Schäffer, H.A. (2003). A view to a post-Boussinesq wave model. Presented at *Coastal Engineering Today*, Gainesville, Florida.

Schäffer, H.A. (2004). Another step towards a post-Boussinesq wave model. Proc. 29th Int. Conf. Coastal Engng. Lisbon, Portugal.

Watson, K.M. and West, B.J. (1975). A transport-equation description of nonlinear ocean surface wave interactions. *J. Fluid Mech*. **70**, 815-826.

West, B.J., Brueckner, K.A., Janda, R.S., Milder D.M. and Milton, R.L (1987). A new numerical method for surface hydrodynamics. *J. Geophys. Res*. **92**(C11), 11803-11824.