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On the use of Conformal Mappings to determine the hydrodynamic force and moment on bodies in viscous incompressible flow.

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1) Introduction

There are at least two reasons to calculate the solution describing the potential flow around a two-dimensional body in translation and rotation. One reason is to determine the corresponding added masses and moment of inertia. The second reason is to use the resulting velocity (actually its gradient) as a basis of functions on which the Navier-Stokes equations –describing an incompressible viscous flow– can be projected. In this context it is known that the calculation of the forces does not require an explicit determination of the pressure

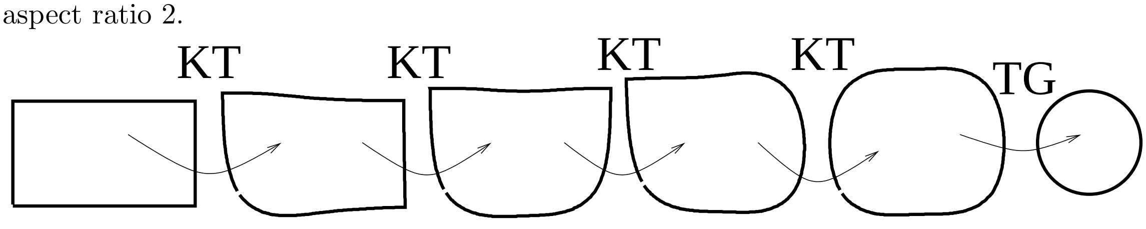
Notwithstanding Newman's opinion (1977, pp 122-123), by using conformal mappings it is shown here that these tasks are considerably simplified. First a classical but simple way to determine the hydrodynamic coefficients is outlined. The obtained numerical results are compared to the quasi-analytical data by Newman (1979) for the square and the finned circle. These shapes are of practical use for studying the energy dissipation associated to rolling motion of ships with knuckles or bilge keels.

On the basis of these developments, the force and moment in incompressible and viscous flow are formulated. Some applications illustrate this formulation.

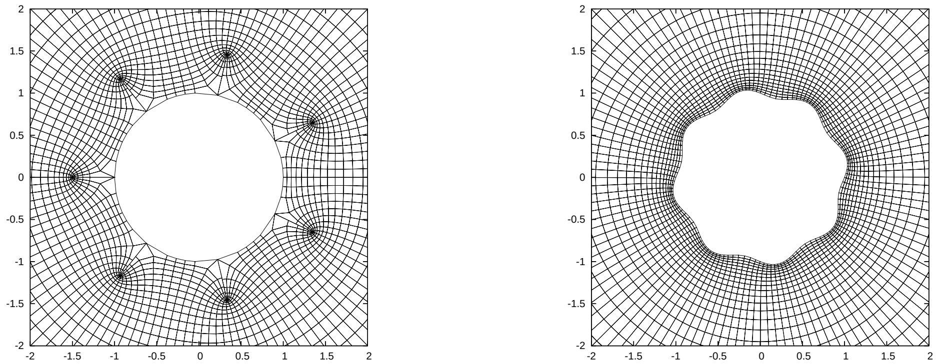
2) Practical context

We consider two-dimensional flow around impermeable (non porous) and rigid bodies. The ambient flow is a time varying current and the body can rotate around a fixed point attached to it. In the restriction of two-dimensional flow around a simply connected body, the conformal mapping technique offers a convenient way to numerically solve the equations and thus describe the flow. When incompressible viscous flows are considered, the conformal mapping first appears as a simple change of variable to transform the more or less complicated fluid domain in a simpler one. In the next developments it is shown that using conformal mapping offers much more than computational simplifications.

The classical transformations and their combinations allow to perform the computations in domain where the image of the body surface is the unit circle. In that case the numerical solver can combine spectral method and other standard techniques. For a rectangular (or a square) section the conformal transformations combine successively the Karmann-Trefftz (KT) transformation (see Halsey, 1979, for example) then the Theodorsen-Garrick (TG) transformation (see Theodorsen and Garrick, 1933). The former transformation can be performed analytically. The latter is more complicated and requires a fixed point algorithm to calculate the mapping function. The following figure illustrates the successive transformations for a rectangle with an aspect ratio 2.



More "exotic" shapes can be considered. For a finned circular section, the conformal mapping is described in Lavrentyev and Chabat (1977). It is a succession of simple transformations of analytical form. From the engineering point of view, it is expected that a two-dimensional model for incompressible viscous flow can provide some information about the loads on a straked cylinder. The "wavy" section described below is also studied for engineering purpose as this is the section of some gravity structures posed on the sea bed. The following figures illustrate these mappings. A regular polar mesh is generated in the unit circle plane then its image is drawn in the physical plane.



In the next developments the mapping function is denoted f. The complex coordinate $z = x + iy = re^{i\theta}$ in the physical plane is related to the complex coordinate $\zeta = \xi + i\eta = \rho e^{i\alpha}$ in the transformed plane by the equation $z = f(\zeta)$. Riemann's Theorem proves that the function f always exists for simply connected bodies.

3) Hydrodynamic coefficients

In order to determine the added masses and the added moment of inertia one has to solve the following three Boundary Value Problems. They are first formulated for the velocity potential

$$\begin{cases} \Delta\phi = 0, & \text{in the fluid domain } \Omega, \\ \vec{\nabla}\phi \cdot \vec{n} = \begin{cases} -\vec{n}\cdot\vec{x}, \\ -\vec{n}\cdot\vec{y}, \\ -\vec{n}\cdot(\vec{z}\wedge\vec{r}), \end{cases} & \text{on the body surface } B, \\ \vec{\nabla}\phi \rightarrow 0, & \text{at infinity.} \end{cases} \tag{1}$$

where $(O,\vec{x},\vec{y},\vec{z})$ denotes a coordinate system with $\vec{z}$ normal to the plane of the flow. The normal $\vec{n}$ is defined on the body surface and the vector $\vec{r} = \vec{OM}$ describes the contour B. The three potentials –noted (ϕ_x,ϕ_y,ϕ_z) corresponding to the three conditions on B– are calculated in the coordinate system attached to the rotating and translating body. Following Milne-Thomson (1969, Art 9.40 pp252–253) or Kochin *et al* (1964, pp 328–329), the same BVPs as (1) can be formulated for the stream function ψ

$$\begin{cases} \Delta\psi = 0, & \text{in the fluid domain } \Omega, \\ \psi = \begin{cases} -y, \\ x, \\ \frac{r^2}{2}, \end{cases} & \text{on the body surface } B, \\ \vec{\nabla}\psi \rightarrow 0, & \text{at infinity.} \end{cases} \tag{2}$$

Then the flow is described by a complex potential $F(z)$ which can be written

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^n} \tag{3}$$

provided that z and ζ are mutual images through the conformal mapping function f. The function F has only poles in the interior of the circle $|\zeta| = 1$, it is hence analytic all over the fluid domain.

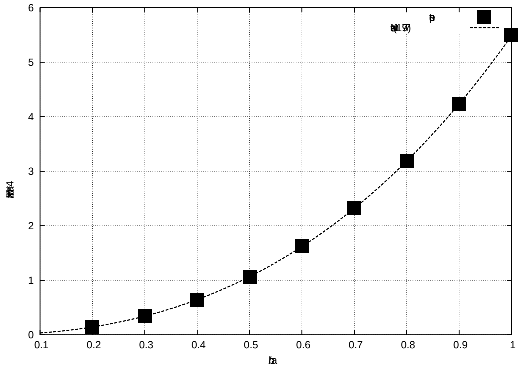
Due to the Dirichlet conditions for ψ on the body contour it is convenient to turn their right hand sides into Fourier series

$$\left.\begin{matrix} -y(\alpha) \\ x(\alpha) \\ \frac{1}{2}r^2(\alpha) \end{matrix}\right\} = \sum_{n=0}^{\infty} A_n \cos n\alpha + B_n \sin n\alpha \tag{4}$$

The identification of $\Im(F)$ with (4) yields the unknown complex coefficients a_n. As a consequence the hydrodynamic coefficients (added masses and added moment of inertia) are deduced from this simple summation

$$\pi\rho_f \sum_{n=0}^{\infty} n(A_n^2 + B_n^2), \tag{5}$$

where ρ_f is the density of the fluid. As an application, the added masses and moment of inertia of the square (with sides $2a$) are computed. It is obtained $M_x = M_y = 4.75376\rho_f a^2$ and $M_z = 0.724579\rho_f a^4$. For the circle with two fins, the analytical expression in Newman (1977, p 145) is compared to the present calculation in the following figure



4) Forces and moment in viscous flows

There exist different ways to compute the forces acting on a solid and impermeable obstacle in an incompressible viscous flow. Basically the Cauchy Stress Tensor must be integrated all over the body contour

$$\vec{F} = -\int_B (p\,\vec{n} + \rho_f \nu\,\omega\,\vec{\tau})\,d\ell \quad \text{(two-dimensional flow).} \tag{6}$$

where ν is the kinematic viscosity, ω denotes the vorticity, p is the pressure and $\vec{\tau}$ is the tangent along the body contour. The pressure usually follows from the integration of its tangential gradient itself deduced from the projection of the Navier Stokes equation on the direction $\vec{\tau}$: $p_{,\tau} = \rho_f \nu \omega_{,n}$. The pressure thus depends on the evaluation of the time rate of circulation on the body.

In the present context an alternative formulation of the force computation is preferred. This formulation was originally proposed by Napolitano et Quartapelle (1983) and it does not require an explicit calculation of the pressure. More recently, Protas *et al.* (2000) presented some applications of this approach. The inner product in the Hilbert space $H_1(\Omega)$ is applied to the pressure p and the potential ϕ. In practice the Navier-Stokes equations is multiplied with $\vec{\nabla}\phi$ and the product is integrated in the fluid domain. After some algebra, it is obtained

$$\frac{1}{\rho_f}\int_B p\,\vec{\nabla}\phi\cdot\vec{n}\,d\ell = -\nu\int_B \vec{\nabla}\phi\cdot(\vec{n}\wedge\vec{\omega})\,d\ell - \int_\Sigma \phi\,\vec{u}_{,t}\cdot\vec{n}\,d\ell - \int_\Omega \vec{\nabla}\phi\cdot(\vec{u}\wedge\vec{\omega})\,ds \tag{7}$$

where Σ is a contour which surrounds the whole fluid domain. In practice this is a circle with a large radius so that the variables on this contour can be matched with the undisturbed flow. In particular the vorticity ω vanishes on this surface. From BVPs (1) ϕ and $\vec{\nabla}\phi\cdot\vec{n}$ are known quantities all over Ω. In particular the boundary conditions on B precisely give the normal gradient of ϕ thus yielding the contribution of the pressure to the hydrodynamic forces and moment

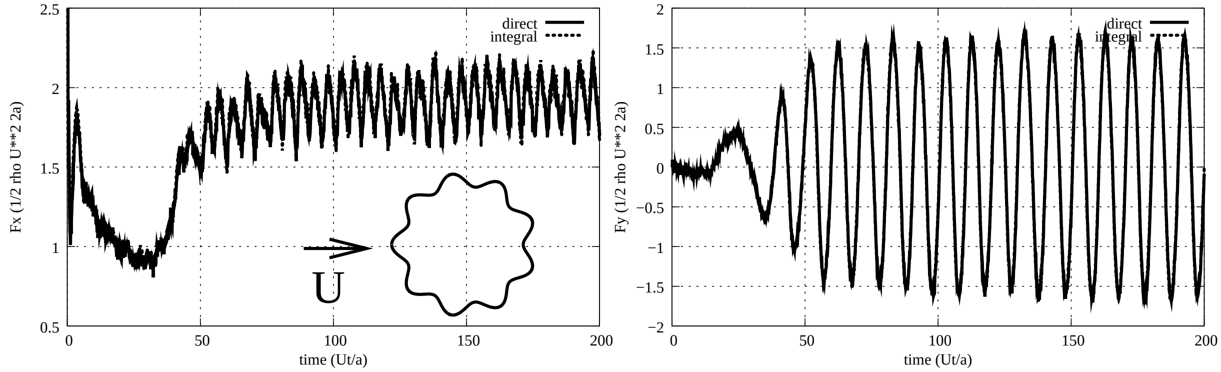
$$\vec{F} = \vec{x}\int_B p\,\vec{\nabla}\phi_x\cdot\vec{n}\,d\ell + \vec{y}\int_B p\,\vec{\nabla}\phi_y\cdot\vec{n}\,d\ell \tag{8}$$

$$\vec{M}_O\cdot\vec{z} = \int_B p\,\vec{\nabla}\phi_z\cdot\vec{n}\,d\ell = \vec{z}\cdot\int_B p\,(\vec{r}\wedge\vec{n})\,d\ell \tag{9}$$

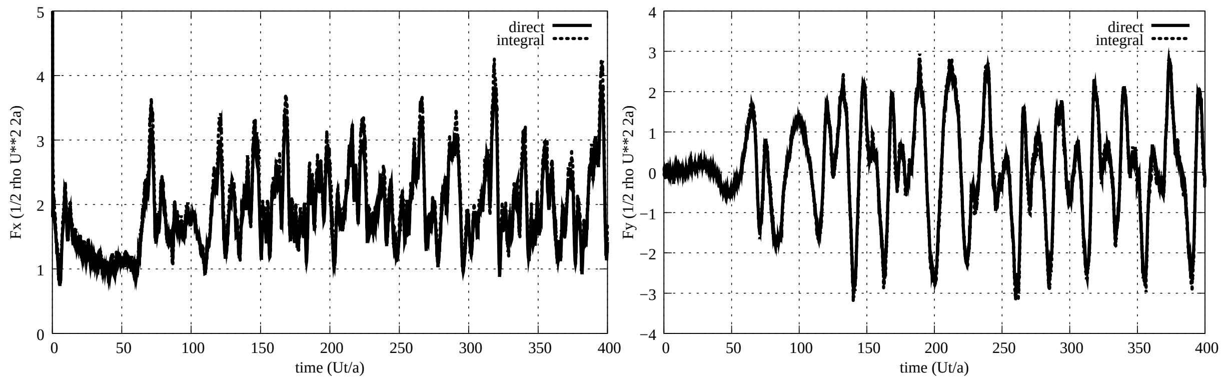
The technique developed in section §3 yields ϕ_z only since the first two variables ϕ_x and ϕ_y are easily obtained analytically

$$\phi_x = -x + \Re\Big[J^\infty\zeta + \frac{\overline{J^\infty}}{\zeta}\Big], \quad \phi_y = -y + \Im\Big[J^\infty\zeta - \frac{\overline{J^\infty}}{\zeta}\Big], \tag{10}$$

where J^{∞} denotes the asymptotic Jacobian $J = dz/d\zeta$ of the transformation at infinity and the overline is the complex conjugate. As an example the following figures show the time variation of the forces (equations 6 and 8 including friction force) acting on a "wavy" contour in a uniform current and Reynolds number $Re = 500$.



These results are obtained with a Vortex-In-Cell Model which solves the two-dimensional Navier-Stokes equations (see Scolan and Faltinsen, 1994). There still remains some numerical difficulties in the present approach. It is known that the derivative of the mapping function (noted $J = dz/d\zeta$) has a singular behavior close to sharp corners. A treatment J^{-1} is necessary and usually it consists in an artificial cut off. The consequences are some discrepancies between the two force formulations. However the agreement is still satisfactory for a square in a uniform current at Reynolds 1000 as shown below.



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