Sloshing Water Waves in a Freely Moving Tank

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1. Setting the Scene:

This work is about water waves which slosh back and forth inside a tank which is free to move horizontally under the action of the waves inside. The tank may also be acted upon by horizontal restoring forces which only respond to the tank's displacement, or the tank may be unconstrained. In this sense the work is motivated by wave loads which are strong enough to overcome fixed constraints and move the container in which the wave is sloshing. Significant wave loads are encountered in the haulage industry when bulk liquids are carried in partially filled tankers, especially when the tankers are manoeuvering. Very large liquid movements, such as suddenly erupting jets, can be made by quite small displacements of the container. The jets can ascend the container walls and damage the roof. So predicting the circumstances under which such jets might occur would be a useful theory.

In the tank which we treat, some possible motions of the system include periodic solutions, in which the tank oscillates with a single frequency ω while waves are sloshing inside the tank and the tank is also moving with frequency ω. The purpose of the work to be presented is to find the mode shapes of the moving free-surface elevation η, the potential $\phi(x, y, t)$ of the underlying velocity field, and the influence of the shape of the tank's boundary on the frequency of oscillation. We concentrate on flow in two space dimensions in tanks with plane walls for which simple expressions for the motion can be found.

This topic is specialised and different from the bulk of past contributions to sloshing. In fixed tanks see for example Evans and Linton (1993). For tanks which are forced to move at prescribed frequencies the conditions at and near resonance have been studied experimentally by Chester and Bones (1968) and theoretically by Chester (1968) and Ockendon *et al.* (1986). Hill and Frandsen (2005) have modelled the growth of water waves under periodic excitation. Ibrahim (2005) exhaustively reviews earlier work on forced, and some examples of free, tank motions, but there remain many theoretically untouched areas in this subject.

2. Analysis of Sloshing:

We take cartesian axes moving with the tank, y increases vertically up and x increases to the right. The still water level lies at $y = 0$. In a laboratory frame (subscript l on variables) with coordinates (x_l, y_l) the tank's centreline has the position $x_l = X(t)$ and the velocity potential is $\phi_l(x_l, y_l, t) = xX' + \phi(x, y, t)$, where prime is the time-derivative. According to linear theory, the velocity potential ϕ in the frame of the tank, satisfies Laplace's equation in the fluid domain below the still water level and the following free-surface boundary conditions:

$$\phi_t + g\eta = -xX'' \qquad \text{on} \qquad y = 0 \tag{1}$$

$$\eta_t = \phi_y \qquad \text{on} \qquad y = 0 \ . \tag{2}$$

Either the bed is a streamline, or equivalently, on the bed $\mathbf{n}$ is a unit normal and

$$\mathbf{n}.\nabla\phi = 0 \ . \tag{3}$$

In order to close the problem we foresee an equation of motion connecting the hydrodynamic forcing on the tank walls with the tank's acceleration. But here we will use an equivalent solvability condition.

3. Some Results:

The presence of a term linear in x on the right-hand side of (1) suggests that both ϕ and η are directly proportional to x. One potential which leads to a simple solution is

$$\phi = x(y+H)f_1(t) \ , \tag{4}$$

where $H > 0$ is a depth. It can be shown that $f_1(t) = A\cos(\omega t + \theta_0)$. The free-surface is plane and one example of a container shape which is compatible with (4), is a V-shaped tank with walls inclined at angle $\pi/4$ to the positive and the negative $x-$axis, and the apex of the V lies at $(x = 0, y = -H)$. This was treated by Cooker (1994) and is in Roberts (2005).

Another solution has a velocity potential directly proportional to the imaginary part of z^4 where the complex variable $z = x + i[y + H]$. A suitable way to write this solution of Laplace's equation is

$$\phi(x, y, t) = (x^3[y + H] - x[y + H]^3)f_3'(t) \ . \tag{5}$$

Equation (2) gives us the form of the corresponding free-surface shape. It is a cubic:

$$\eta(x, t) = (x^3 - 3H^2x)f_3(t) \ , \tag{6}$$

where the time-integration constant vanishes to ensure that $y = 0$ is the mean water level. Both ϕ and η are odd with respect to x, in order to ensure a coupling between the liquid motion and the sideways oscillation of the tank. The coefficients of x^3 and x which come out of equation (1) give us two ordinary differential equations, for $f_3(t)$ and $X(t)$:

$$f_3'' + \frac{g}{H}f_3 = 0 \qquad \text{and} \qquad X'' = -H^3f_3'' - 3gH^2f_3 \ . \tag{7}$$

The first of these gives $f_3(t) = A\cos(\omega t + \theta_0)$, where $\omega^2 = g/H$ and θ_0 is a phase constant; and this completes the description of the wave elevation in equation (6). The second member of equation (7) dictates that for these waves to occur, the tank must have a motion given by

$$X(t) = -2gH^2A\cos(\omega t + \theta_0) \ . \tag{8}$$

The oscillation frequency ω is compatible with the wave forcing on the walls of the tank. The liquid force on the tank also has frequency ω. We can satisfy Newton's second law for the motion of the tank, (in the laboratory frame) $mX'' + rX = k\cos(\omega t + \theta_0)$, where m is the mass of the tank, r is the coefficient of any restoring force and k is a constant which depends on the geometry of the tank.

Now we find the shape of the tank. With respect to the tank's frame of reference, the stream function associated with equation (5) is directly proportional to the real part of z^4:

$$\psi(x, y, t) = (x^4 - 6x^2[y + H]^2 + [y + H]^2)f_3'(t)/4 \ .$$

(In the laboratory frame of reference the stream function ψ_l includes the oscillation of the tank, and it is $\psi_l = \psi + yf_3'$.) Now in order to satisfy condition (3) ψ is constant (spatially uniform) on certain straight lines ($y + H = x\tan\alpha$) which pass through $(x = 0, y = -H)$ at angles α of elevation $\pi/8$, $3\pi/8$, $5\pi/8$ and $7\pi/8$ to the positive $x-$axis. These lines (and other streamlines defined by ψ) can be used to form examples of shapes for the bed of the tank, consistent with the above velocity field, free-surface and tank motions.

4. Example and Discussion:

The tank defined by the streamlines with angles $\alpha = \pi/8$ and $\alpha = 3\pi/8$, holds fluid with a still waterline which lies at $y = 0$ between $x = H\tan(\pi/8) = 0.414H$ and $x = H\tan(3\pi/8) = 2.41H$. Between these limits the free surface oscillates with a node at $x = H\sqrt{3} = 1.73H$, where $\eta = 0$ for all time. The expression for $\eta(x,t)$ given by (6) conserves the volume of fluid. An interesting feature of the wave is that the free-surface contact point on the right-hand slope has much greater extremes in elevation in its excursion up and down the beach than does the left-hand contact point on its side where the wall overhangs the fluid domain. Also the right-hand contact point has an amplitude of the horizontal component of oscillation which is much greater than the amplitude of the horizontal oscillation of the tank. This is an example of large displacement of a liquid jet ascending a fluid boundary, induced by a relatively small normal component of displacement of the boundary. This may help explain the hazardous liquid jets which shoot out of carelessly handled containers. Two applications of these preliminary results may bear on the handling of heavy liquids poured from light containers and in considering the generation of water waves by the seismic displacement of the sea bed near beaches.

Other solutions will be discussed at the workshop. As suggested in the sentence before equation (5), one can construct other velocity potentials, ϕ, from the imaginary part of $(x + i[y + H])^n$, where the power n is 2 or 4 in this abstract, but n can be any even integer. The corresponding surface elevations for $n = 6, 8, 10, ...$ are polynomials in x of degrees $5, 7, 9, ...$ The corresponding containers are sectors bounded by a selection from n planes whose adjacent slopes subtend angle π/n at $(0, -H)$.

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