

An unstructured DG-FEM method for nonlinear wave-structure interaction

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This abstract describes recent work on applying the Discontinuous Galerkin Finite Element Method (DG-FEM) to solve a set of highly accurate Boussinesq-type equations for the interaction of nonlinear waves with bottom mounted structures. We present here some validation results in 2-D which establish the accuracy and convergence properties of the scheme, along with some preliminary results in 3-D for the diffraction of waves around bottom mounted structures.

The mathematical model being applied is derived in [7, 6], and we consider both the Padé (4,4) and the Padé (2,2) versions of this method which can accurately propagate nonlinear waves in relative water depths up to $kh \approx 25$ & 10 respectively, (k the wavenumber, h the water depth) up to something near the stable breaking limit. The associated wave kinematics (vertical distribution of pressure and velocity) are accurate up to $kh \approx 12$ & 4 respectively. A finite difference based solution to these equations has been used to obtain solutions for highly nonlinear wave run-up on piecewise rectangular structures as described in [4, 5] and references therein. This solution, as implemented, is limited to a rectangular, uniformly spaced grid and hence piecewise rectangular structures. The main goal of the present work is to remove this limitation and allow the treatment of more general geometries. The choice of a DG-FEM methodology provides the geometric flexibility of an unstructured grid while also providing flexibility in the local order of accuracy of the numerical scheme. This combination could lead to improved computational efficiency, although this has yet to be demonstrated.

To briefly outline the mathematical model, we consider water waves in 3-D using a coordinate system with origin in the still-water plane and the z -axis vertically upwards. The fluid domain is bounded by the sea bed at $z = -h(\mathbf{x})$, with $\mathbf{x} = [x, y]$, and the free surface at $z = \eta(\mathbf{x}, t)$, where t is time. The free surface boundary conditions are written in terms of velocity components at the free surface $\tilde{\mathbf{u}} = [\tilde{u}, \tilde{v}] = \mathbf{u}(\mathbf{x}, \eta, t)$ and $\tilde{w} = w(\mathbf{x}, \eta, t)$:

$$\frac{\partial \eta}{\partial t} = \tilde{w} - \nabla \eta \cdot (\tilde{\mathbf{U}} - \tilde{w} \nabla \eta), \quad (1)$$

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} = -g \nabla \eta - \frac{1}{2} \nabla (\tilde{\mathbf{U}} \cdot \tilde{\mathbf{U}} - \tilde{w}^2 (1 + \nabla \eta \cdot \nabla \eta)), \quad (2)$$

where $\nabla = [\partial/\partial x, \partial/\partial y]$ is the 2-D gradient operator and

$$\tilde{\mathbf{U}} = [\tilde{U}, \tilde{V}] = \tilde{\mathbf{u}} + \tilde{w} \nabla \eta. \quad (3)$$

Evolving η and $\tilde{\mathbf{U}}$ forward in time requires a means of computing \tilde{w} subject to the Laplace equation and the kinematic bottom condition

$$w + \nabla h \cdot \mathbf{u} = 0, \quad z = -h. \quad (4)$$

A highly-accurate Boussinesq method for this purpose is obtained by approximating the vertical distribution of fluid velocity by

$$\mathbf{u}(\mathbf{x}, z, t) = (1 - \alpha_2 \nabla^2 + \alpha_4 \nabla^4) \hat{\mathbf{u}}^*(\mathbf{x}, t) + ((z - \hat{z}) \nabla - \beta_3 \nabla^3 + \beta_5 \nabla^5) \hat{w}^*(\mathbf{x}, t), \quad (5)$$

$$w(\mathbf{x}, z, t) = (1 - \alpha_2 \nabla^2 + \alpha_4 \nabla^4) \hat{w}^*(\mathbf{x}, t) - ((z - \hat{z}) \nabla - \beta_3 \nabla^3 + \beta_5 \nabla^5) \hat{\mathbf{u}}^*(\mathbf{x}, t), \quad (6)$$

The quantities $\hat{\mathbf{u}}^*$ and \hat{w}^* are utility variables which have been introduced to allow Padé enhancement of the Taylor series operators. Optimal velocity distributions are obtained near $\hat{z} = -h/2$, and we adopt this value here. The coefficients α_i, β_i are polynomials in $(z - \hat{z})$ and depend on whether the Padé (4,4) or (2,2) form of the method is chosen. Inserting (5) and (6) into (4) gives one equation relating $\hat{\mathbf{u}}^*$ and \hat{w}^* to each other. Combining this with (5) applied at $z = \eta$, while also invoking (3) gives a 3×3 system that can be solved for $\hat{\mathbf{u}}^*$ and \hat{w}^* in terms of $\tilde{\mathbf{U}}$ and η . The resulting system appears in matrix form as

$$\begin{bmatrix} \mathcal{A}_1 - \eta_x \mathcal{B}_{11} & -\eta_x \mathcal{B}_{12} & \mathcal{B}_{11} + \eta_x \mathcal{A}_1 \\ -\eta_y \mathcal{B}_{11} & \mathcal{A}_1 - \eta_y \mathcal{B}_{12} & \mathcal{B}_{12} + \eta_y \mathcal{A}_1 \\ \mathcal{A}_{01} & \mathcal{A}_{02} & \mathcal{B}_0 \end{bmatrix} \begin{bmatrix} \hat{u}^* \\ \hat{v}^* \\ \hat{w}^* \end{bmatrix} = \begin{bmatrix} \tilde{U} \\ \tilde{V} \\ 0 \end{bmatrix}, \quad (7)$$

where we have also assumed the flow to be irrotational. (A slightly more complicated form results if we allow vorticity in the horizontal plane). Here the subscripts x and y denote partial differentiation. Having solved (7) for the utility variables, equation (6) invoked at $z = \eta$ closes the problem. In 2-D, (7) reduces in the obvious way to a 2×2 system of equations. The full expressions for the coefficients α_i, β_i and the differential operators \mathcal{A}_{ij} and \mathcal{B}_{ij} are omitted here for brevity, but can be found in the above mentioned references. Note however that in Padé (4,4) form the operators involve up to fifth partial derivatives in x and y , while in Padé (2,2) form up to third derivatives are required.

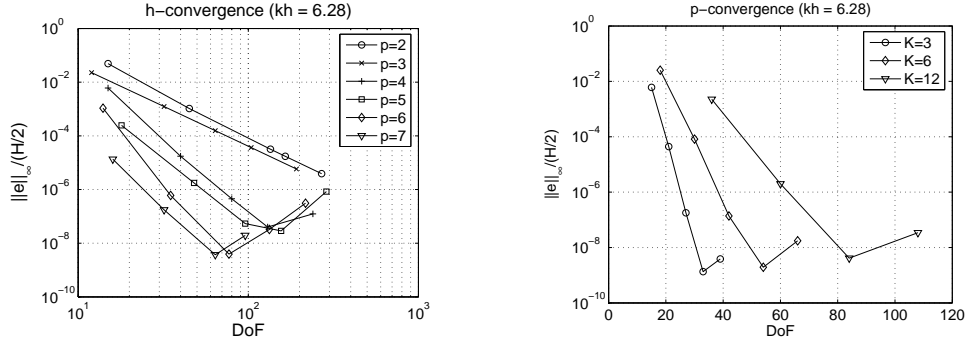
To obtain a numerical solution we adopt a method of lines approach and choose an explicit Runge-Kutta scheme for the time evolution of the free-surface boundary conditions (1) & (2). The discretisation of spacial derivatives appearing in these equations and in (7) & (6) is performed using a DG-FEM on an unstructured grid of non-overlapping triangles distributed over the still water plane. The DG-FEM method was developed for solving conservation laws, (see the review by [1]), and it is a Galerkin-type finite-element method which achieves spectral convergence as the order of the basis is increased. The method is local in nature, with the solution allowed to be discontinuous between elements which are coupled by imposing the finite-volume-like idea of numerical fluxes. It's application to the above Boussinesq equations will be briefly described here, further details can be found in [3, 2]. The domain is partitioned into K elements, and on the k 'th element the solution is approximated by a finite series as $u^k(\mathbf{x}) = \sum_{i=0}^N \hat{u}_i^k \phi_i(\mathbf{x})$. Where \hat{u}_i^k is a modal amplitude and ϕ_i is the i 'th polynomial basis function of at most order p from a complete orthonormal set. The equations are then multiplied by the same set of basis functions (a Galerkin method) and integrated over the element to give *e.g.* for the free-surface conditions

$$\int_{\Omega_k} \phi_i \frac{\partial \eta}{\partial t} d\mathbf{x}^k = \int_{\Omega_k} \phi_i G(\mathbf{x}) d\mathbf{x}^k \quad (8)$$

$$\int_{\Omega_k} \phi_i \frac{\partial \tilde{\mathbf{U}}}{\partial t} d\mathbf{x}^k = - \int_{\Omega_k} \phi_i \nabla F(\mathbf{x}) d\mathbf{x}^k + \oint_{\partial\Omega_k} \phi_i \mathbf{n} \cdot (F(\mathbf{x}) - F^*(\mathbf{x})) d\mathbf{x}^k \quad (9)$$

where $i = 0, 1, \dots, N$. Here Ω_k is the volume of the k 'th element with $\partial\Omega_k$ its boundary surface and \mathbf{n} the outward pointing normal; G & $-\nabla F$ are the right hand sides of (1,2).

The boundary integral in (9) is obtained from two integration by parts with the exact flux F replaced by a numerical flux F^* after the first integration. This numerical flux is used to couple adjacent elements. Replacing the continuous variables with their modal expansions, and choosing suitable numerical fluxes leads to a discrete numerical method. Similarly, for the discrete versions of (6,7) each high-order differential operator is decomposed into a succession of first derivatives applied to utility variables which leads to expressions in terms of matrix-matrix products of first-order derivatives. Central fluxes are chosen for the derivative operators while Lax-Friedrichs fluxes are used in (9) in order to introduce a small amount of numerical diffusion.



a) Δx -type refinement.

b) p -type refinement.

Figure 1: Convergence of the solution for the linear problem.

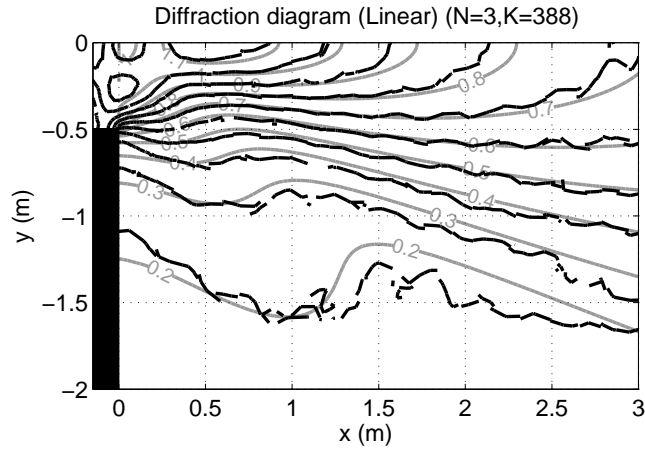


Figure 2: Contour plots for linear diffraction around a semi-infinite breakwater $kh = 1.3$. Linear theory (grey), calculations (black).

The performance of the method in 2-D is tested using a linear standing wave and Figure 1 shows the convergence of the calculations under both grid-refinement and refinement in polynomial order. Algebraic convergence is obtained under grid-refinement and is found to be $O(p + 1)$ for even polynomial orders and $O(p)$ for odd polynomial orders. Exponential convergence is found under refinement of the basis polynomial order.

Figures 2 & 3 show calculations in 3-D for linear diffraction around a semi-infinite breakwater and around a bottom mounted circular cylinder in a channel. Further results will be presented at the workshop.

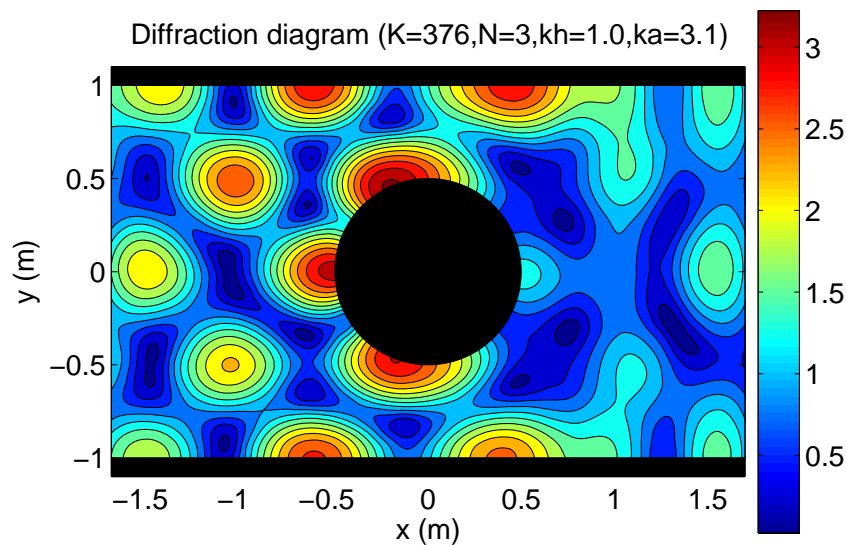


Figure 3: Linear diffraction around a cylinder of radius $a = 0.5m$ in a channel.

References

- [1] B. Cockburn and C.-W. Shu. Runge-Kutta discontinuous Galerkin methods for convection-dominated problems. *J. Scientific Computing*, 16(3):173–261, 2001.
- [2] A.P. Engsig-Karup, J.S. Hesthaven, H.B. Bingham, and P.A. Madsen. Nodal DG-FEM solutions of high-order Boussinesq-type equations. *J. Engineering Math.*, To appear, 2006.
- [3] C. Eskilsson, A. P. Engsig-Karup, S. J. Sherwin, J. S. Hesthaven, and L. Bergdahl. The next step in coastal numerical models: spectral/hp element methods? In *Proceedings 5th international symposium on ocean wave measurement and analysis (WAVES2005)*, Madrid, Spain, 2005.
- [4] D.R. Fuhrman, H.B. Bingham, and P.A. Madsen. Nonlinear wave-structure interaction with a high-order Boussinesq model. *Coastal Engineering*, 52:655–672, 2005.
- [5] E. Jamois, D.R. Fuhrman, H.B. Bingham, and B. Molin. Wave-structure interactions and nonlinear wave processes on the weather side of reflective structures. *Coastal Engineering*, Submitted.
- [6] P.A. Madsen, H.B. Bingham, and H. Liu. A new Boussinesq method for fully nonlinear waves from shallow to deep water. *J. Fluid Mech.*, 462:1–30, 2002.
- [7] P.A. Madsen, H.B. Bingham, and H.A. Schäffer. Boussinesq-type formulations for fully nonlinear and extremely dispersive water waves: Derivation and analysis. *Proc. R. Soc. Lond. A*, 459:1075–1104, 2003.