Wave diffraction by a periodically constrained elastic plate floating on water

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Introduction

In this contribution we consider how flexural-gravity waves propagating on a thin elastic plate in contact with an ideal fluid of constant finite depth interact with points which are constrained to move via an impedance condition. The problem is designed to model a large floating offshore platform which is either supported rigidly by a number of fixed columns to the sea bed, or tethered by extensible mooring lines. Such problems may be of interest in the design of large floating structures currently being considered and tested for use as floating offshore runways and oil storage facilities. The work described here is an extension of the work presented at the last workshop by Evans & Meylan (2005).

Formulation of the problem

At rest, a thin elastic plate occupies the x, y plane with its lower surface coinciding with $z = 0$ and in contact with an ideal fluid which extends vertically through the depth $-h < z < 0$.

Linearised theory is used to describe the small-amplitude fluid motion which is defined by a velocity potential $\Phi(\mathbf{r}, z, t)$ with $\mathbf{r} = (x, y)$. Assuming time-harmonic motions of angular frequency ω allows us to write $\Phi(\mathbf{r}, z, t) = \Re\{-i\omega\phi(\mathbf{r}, z)\mathrm{e}^{-i\omega t}\}$. Within the fluid Laplace's equation is satisfied,

$$\left(\Delta + \partial^2/\partial z^2\right)\phi = 0, \qquad -h < z < 0, \ -\infty < x, y < \infty \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ with the sea-bed condition $\partial\phi/\partial z|_{z=-h} = 0$. The elastic plate has a displacement given by $\Re\{u(\mathbf{r})\mathrm{e}^{-i\omega t}\}$ which is connected to ϕ via the linearised kinematic condition

$$\partial\phi/\partial z|_{z=0} = u(\mathbf{r}) \tag{2}$$

and through the time-harmonic equation of motion of the plate (modelled by Kirchhoff theory),

$$(D\Delta^2 - m\omega^2)u(\mathbf{r}) = [p] = \rho_w\omega^2\phi(\mathbf{r}, 0) - \rho_w g u(\mathbf{r}) \tag{3}$$

after using the linearised Bernoulli equation for the jump in pressure across the plate, $[p] \equiv [p]_{z=0^+}^{z=0^-}$. In the left-hand side of (3), $D = Ed^3/(12(1-\nu^2))$ is the flexural rigidity of the plate, expressed in terms of its Young's modulus, E, the thickness d and Poisson's ratio ν, whilst $m = \rho_p d$ is the mass per unit area on the plate, expressed in terms of the plate density, ρ_p. On the right-hand side of (3), ρ_w is the water density and g is gravitational acceleration.

Combining (2) and (3) to eliminate $u(\mathbf{r})$ and then non-dimensionalising variables (e.g. Williams & Squire (2004)) allows the equation on $z = 0$ to be written in terms of a single dimensionless parameter

$$(\Delta^2 + \varpi)\partial\phi/\partial z|_{z=0} - \phi(\mathbf{r}, 0) = 0. \tag{4}$$

with

$$\varpi = \omega^{*-8/5}(1 - m^*\omega^{*2}), \qquad m^* = \frac{\rho_p d}{\rho_w L}, \quad \omega^* = (L/g)^{1/2}\omega$$

and $L = (D/\rho_w g)^{1/4}$, whilst lengths have been scaled by $L' = L\omega^{*-2/5}$. Equations (1), (2) remain unchanged by the non-dimensionalisation. The parameter ϖ captures both the frequency of motion and the competition between inertia and hydrodynamic forces. For example, when $\varpi < 0$, the effect of the water is negligible and, conversely, for large ϖ the inertial effects of the plate can be ignored. The non-dimensional depth, $h^* = h/L'$ is such that $h^* < \frac{1}{2}$ is well-approximated by shallow water theory and $h^* > 4$ is effectively infinite depth.

In the presence of wave motion on the elastic plate, the constraint of a single point implies the application of a time-harmonic point force, F say, at that point (the origin say) which is included as an extra pressure term $F\delta(x)\delta(y)$ on the right-hand side of (3). The resulting non-dimensionalisation of the plate equation gives rise to a dimensionless force $F^* = F\omega^{*-4/5}/(\rho_w g L^3)$ and (4) is replaced by

$$(\Delta^2 + \varpi)u(\mathbf{r}) - \phi(\mathbf{r}, 0) = F^*\delta(x)\delta(y).$$

If the point is to be fixed, then F^* is determined by application of $u(0,0) = 0$. If the point is constrained to move by the attachment of a point mass M, and a spring of spring constant κ then $F = (M\omega^2 - \kappa)u(0,0)$. In dimensionless variables, $F^* = \mu u(0,0)$, where $\mu = \omega^{*-4/5}(M^*\omega^{*2} - \kappa^*)$ and $M^* = M/(\rho_w L^3)$ and $\kappa^* = \kappa/\sqrt{\rho_w g D}$ are dimensionless mass and spring constants. Letting $|\mu| \to \infty$ has the same effect as holding the point fixed. Henceforth, asterisks will be dropped.

The solution to the problem of the scattering of an incident wave (described by the potential ϕ_i and having displacement u_i) by a single constrained point, is given by

$$\phi(\mathbf{r}, z) = \phi_i(\mathbf{r}, z) + FG(\mathbf{r}, z; 0, 0), \qquad \text{implying} \qquad u(\mathbf{r}) = u_i(\mathbf{r}) + Fg(\mathbf{r}; 0, 0) \tag{5}$$

where $g = \partial G/\partial z|_{z=0}$ and $G(\mathbf{r}, z; \mathbf{r}')$ is the Green function for a plate over water, satisfying

$$(\Delta^2 + \varpi)g(\mathbf{r}; \mathbf{r}') - G(\mathbf{r}, 0; \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

in addition to Laplace's equation and the bottom condition. It can be shown that

$$g(\mathbf{r}; \mathbf{r}') = \frac{i}{4} \sum_{m=-2}^{\infty} \tau_m H_0^{(1)}(k_m \rho), \qquad \tau_m = \frac{[Y_m'(0)]^2}{C_m} \tag{6}$$

where $H_0^{(1)}(z)$ is the first-kind Hankel function, $\rho = |\mathbf{r} - \mathbf{r}'|$, and $Y_m(z) = \cosh k_m(z+h)$ are depth eigenfunctions which arise when considering separation solutions. Also $C_m = \frac{1}{2}[h + (5k_m^4 + \varpi)\sinh^2 k_m h]$, whilst k_m ($m \geq -2$) are the roots of the dispersion relation

$$(k_m^4 + \varpi)k_m \tanh k_m h - 1 = 0. \tag{7}$$

We define k_0 as the unique positive real root of (7), representing travelling waves of wavelength $\lambda = 2\pi/k_0$, whilst k_m, $m \geq 1$ represent the sequence of pure imaginary roots with positive (and increasing with m) imaginary parts. Two generally complex roots exist in the upper-half plane and are denoted by k_{-1} and $k_{-2} = -\overline{k_{-1}}$.

The displacement due to the incident wave, introduced in (5), travelling at an angle ψ to the positive x-axis, can now be written $u_i(\mathbf{r}) = \exp\{ik_0(x\cos\psi + y\sin\psi)\}$.

Despite the apparent log-singularity in g in (6), various relations, including crucially that $\sum_m \tau_m = 0$, can be used to show that the Green function is bounded at the origin. In fact, as $\rho \to 0$,

$$g(\mathbf{r}; \mathbf{r}') \sim C + \frac{\rho^2 \log \rho}{8\pi} + O(\rho^2), \qquad \text{where} \qquad C = -\frac{1}{2\pi} \sum_{m=-2}^{\infty} \tau_m \ln k_m. \tag{8}$$

Using (8) in (5) with the impedance relation $F = \mu u(0,0)$ gives $u(0,0) = 1 + \mu u(0,0)C$ which solves to give $u(0,0) = 1/(1 - C\mu)$ and hence the plate displacement everywhere is given by

$$u(\mathbf{r}) = u_i(\mathbf{r}) + \frac{\mu}{1 - C\mu} g(\mathbf{r}; 0, 0).$$

Diffraction of incident waves by an arbitrary arrangement of points

The extension from the scattering by a single point at the origin to the scattering by N points located at $\mathbf{r} = \mathbf{r}_n$ on the plate and constrained to move via an impedance μ_n is trivial, the general solution being given by superposition,

$$u(\mathbf{r}) = u_i(\mathbf{r}) + \sum_{n=1}^{N} F_n g(\mathbf{r}; \mathbf{r}_n) \tag{9}$$

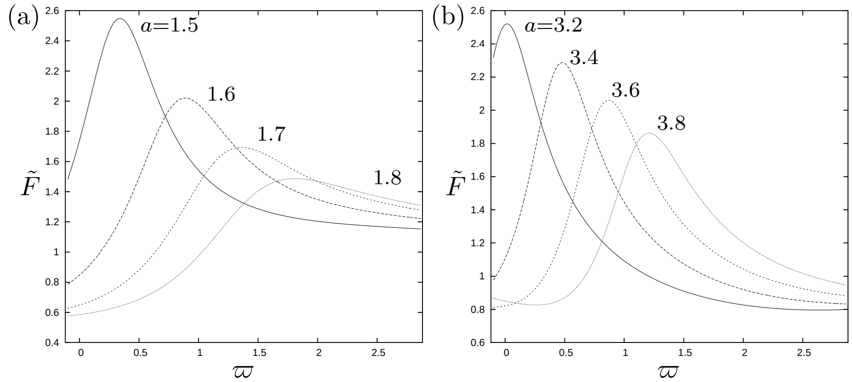

Figure 1: Non-dimensional force on the pinned point at $(-a, a)$ against ϖ with $h = 1$ and $\psi = 45°$

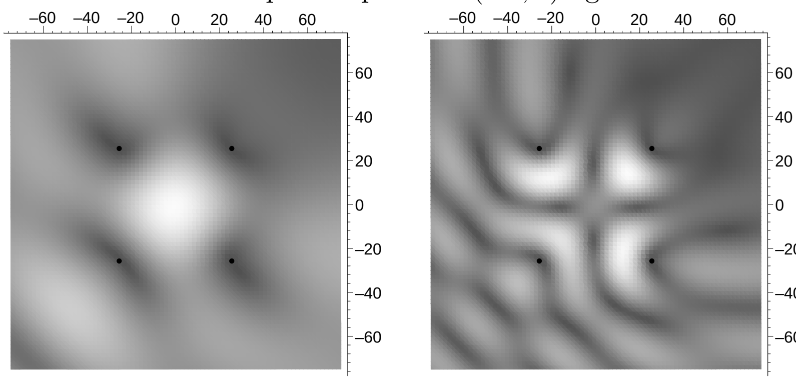

Figure 2: An example of maximum amplitude of displacement on the plate at near-resonance.

The external force at the nth pin is F_n which, by definition, satisfies $F_n = \mu_n u(\mathbf{r}_n)$, $n = 1, 2, \ldots, N$ and so we obtain the system of equations

$$F_m = \mu_m u_i(\mathbf{r}_m) + \mu_m \sum_{n=1}^{N} F_n g(\mathbf{r}_m; \mathbf{r}_n), \quad m = 1, 2, \ldots, N$$

that determine F_m. A good example of the interesting results that can be found is given by choosing $N = 4$ fixed points ($\mu_n^{-1} = 0$) arranged at the vertices of a square ($\mathbf{r}_n = (\pm a, \pm a)$, say). Figure 1 shows the force (non-dimensionalised w.r.t. the force on an isolated pinned point) on one of the points as ϖ varies for $\psi = 45°$, with $h = 1$ and for a set of values of a. In terms of dimensional quantities, the choice $E = 5$GPa, $\rho_p = 922.5\text{kg/m}^3$, $\rho_w = 1025\text{kg/m}^3$, $g = 9.81\text{m/s}^2$, $\nu = 0.3$, $d = 1$m, gives dimensional values (in metres) of point separation, water depth and wavelength at each of the four peaks in figure 1(a) as (34, 11, 70), (44, 14, 93), (52,15,112), (58,16,125) and in figure 1(b), as (54,9,52), (84,14,82), (104,14,100), (118,15,115). The peak in the force is associated with a near-resonance in which the wavelength is approximately half – or equal to – the spacing of the points (see figure 2).

Infinite periodic arrays of points

For large, but finite regular arrays of points, it proves useful to consider problems involving infinite periodic arrays, where further analytic progress is possible. Following (9) we write, for a single periodic array of points,

$$u(\mathbf{r}) = u_i(\mathbf{r}) + \sum_{n=-\infty}^{\infty} F_n g(\mathbf{r}; \mathbf{r}_n), \qquad \text{where } F_n = \mu u(\mathbf{r}_n) \text{ and } \mathbf{r}_n = (na, 0),\ n \in \mathbb{Z}. \tag{10}$$

The imposed periodicity in x allows us to invoke Floquet theory which states that there can only be a change in phase in the solution, being equal to that of the incident wave, across successive periods. Hence, $F_n = F_{n-1}\sigma$, where $\sigma = \mathrm{e}^{i\alpha_0 a}$ and $\alpha_0 = k_0 \cos\psi$ and it follows that $F_n = \sigma^n F_0$. Then, from (10) with $\mathbf{r} = \mathbf{r}_m$, $m \in \mathbb{Z}$ it is not difficult to show that that

$$F_0 = \mu/(1 - \mu S), \quad \text{where} \quad S = \sum_{n=-\infty}^{\infty} \sigma^{-n} g(na, 0; 0, 0).$$

In its present form with g given by (6), the series for S converges slowly, but use of an integral representation of the Hankel function and Poisson's summation formula allows us to write S as

$$S(k_0,\alpha_0,a)=\frac{1}{2a}\sum_{n=-\infty}^{\infty}\sum_{m=-2}^{\infty}\frac{\tau_m}{\lambda_{mn}},\qquad \text{where } \lambda_{mn}^2=\alpha_n^2-k_m^2,\quad \alpha_n=\alpha_0+2n\pi/a. \tag{11}$$

This approach allows us to consider scattering of waves by an infinite periodic array of points, and determine the reflection and transmission coefficients. It turns out that under certain circumstances (examples will be given at the Workshop) *total* reflection from the infinite array is possible.

Alternatively, one can consider so-called Rayleigh-Bloch waves in which the incident wave is absent, and the frequency operates in a range where radiation of energy from the array is prohibited. The resulting waves are local to the array and determined simply by satisfying

$$1/\mu=S(k_0,\alpha_0,a) \tag{12}$$

with $\alpha_0>k_0$ ensuring that this is a real condition. It is necessary to consider only values of $\alpha_0\in[0,\pi/a]$ since, for $m\in\mathbb{Z}$, $S(k_0,2m\pi/a+\alpha_0,a)=S(k_0,\alpha_0,a)=S(k_0,2\pi/a-\alpha_0,a)$. Interestingly, in order to obtain solutions of (12), we must have $\mu^{-1}\neq 0$, implying that pinned points do not support Rayleigh-Bloch waves. Also, the presence of Rayleigh-Bloch waves leads to large near-resonant motions near the centre of a long finite array of constrained points, in much the same way as observed by Maniar & Newman (1997) for long arrays of cylinders excited by surface gravity waves on water.

Perhaps a more realistic situation is one where the elastic plate is constrained to move on a finite rectangular grid. It now proves instructive to consider a doubly-infinite periodic array of points on an elastic plate, each constrained to move by the same impedance condition. There can be no incident wave for such a problem and it makes sense only to consider if waves are able to propagate without attenuation throughout the array. That is, we seek pass-band lattice modes (well-known in other wave theories, such as optics, crystallography etc). The constraints are placed on the rectangular grid $\mathbf{r}=\mathbf{r_p}=(pa,qb)$, $\mathbf{p}=(p,q)\in\mathbb{Z}^2$. Applying Bloch-Floquet theory implies that for $\mathbf{r}\in\mathbb{R}^2$,

$$u(\mathbf{r}+\mathbf{r_p})=u(\mathbf{r})\exp(i\boldsymbol{\alpha}.\mathbf{r_p}),\qquad \mathbf{p}=(p,q)\in\mathbb{Z}^2 \tag{13}$$

where $\boldsymbol{\alpha}=(\alpha_0,\beta_0)$ and α_0, β_0 are Bloch wavenumbers in the x, y directions respectively. It follows that

$$u(\mathbf{r})=\sum_{\mathbf{q}}g(\mathbf{r};\mathbf{r_q})\exp(i\boldsymbol{\alpha}.\mathbf{r_q}),$$

satisfies (13) where $\mathbf{q}=(r,s)\in\mathbb{Z}^2$ and $g(\mathbf{r};\mathbf{r}')$ is the Green function defined by (6). Application of the impedance condition at each point implies $u(\mathbf{r_p})=u(0,0)\exp(i\boldsymbol{\alpha}.\mathbf{r_p})/\mu$ whilst $u(0,0)=1$ can be chosen without loss of generality, whence

$$1/\mu=S,\qquad \text{where}\qquad S=\sum_{\mathbf{p}}g(\mathbf{r_p};0)\exp(-i\boldsymbol{\alpha}.\mathbf{r_p}).$$

Again, integral representations and Poisson's formula leads us to the rapidly convergent expression

$$S=\frac{1}{2a}\sum_{p=-\infty}^{\infty}\sum_{m=-2}^{\infty}\tau_m\frac{\sinh\lambda_{mp}b}{\lambda_{mp}(\cosh\lambda_{mp}b-\cos\beta_0 b)},$$

where λ_{mp} and α_p are defined by (11). It should be noted that S is always real-valued for all values of k_0. Results showing the location of pass-bands and stop-bands and their effect upon finite arrays will be presented at the Workshop.

References

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Williams, T. & Squire, V. 2004 Oblique scattering of plane flexural-gravity waves by heterogenities in sea-ice. *Proc. Roy. Soc. Lond. A.* **460**, 3469–3497.

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Discusser - M.H. Meylan:

Why is this problem simpler than bottom mounted cylinders?

Reply:

Each scatterer is a single point on the elastic plate, as opposed to a cylinder of non-zero radius. This means that the contribution to scattering by each point is a single Green function. The analysis for multiple scattering thus becomes as simple as one can imagine, but many of the features displayed by scattering on elastic plates are shared by scattering by cylinders which makes elastic plate problems a useful tool.

Discusser - R. W. Yeung

This is a very compact analysis of a plate and fluid interaction. What is not too obvious is the source of such plane elastic waves. If the elastic plate is excited by ocean waves, the incident wave from the plate edges would not be planar.

Reply:

In reality, there may be some boundary between the free surface of the fluid and the elastic plate. Then the flexural waves on the elastic plate will have originated from the ocean waves. The current analysis owes its elegance and simplicity to the fact that we neglect the boundary of the elastic plate and simply consider local wave interactions from the points that support the plate. Any attempt to include the finite extent of the elastic plate would almost inevitably require a numerical approach and the key features associated with the supporting points would be harder to analyse.