Free-surface Wave Interaction with a Thick Flexible Platform

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Introduction

We consider the two-dimensional interaction of an incident wave with a flexible floating dock or very large floating platform (VLFP) with finite draft. The water depth is finite. The case of a rigid dock is a classical problem. For instance Mei and Black have solved the rigid problem, by means of a variational approach. They considered a fixed bottom and fixed free surface obstacle, so they also covered the case of small draft. After splitting the problem in a symmetric and an antisymmetric one the method consists of matching of eigenfunction expansions of the velocity potential and its normal derivative at the boundaries of two regions. In principle, their method can be extended to the flexible platform case. Recently we derived a simpler method for both the moving rigid and the flexible dock. However we considered objects with zero draft only. In this paper we extend our approach to the case of finite, but small, draft. The draft is small compared to the length of the platform to be sure that we may use as a model, for the elastic plate, the thin plate theory, while the water pressure at the plate is applied at finite depth. The method is based on a direct application of Green's theorem, combined with an appropriate choice of expansion functions for the potential in the fluid region outside the platform and the deflection of the plate. The integral equation obtained by the Green's theorem is transformed in an integral-differential equation by making use of the equation for the elastic plate deflection. One must be careful in choosing the appropriate Green's function. It is crucial to use a formulation of the Green's function consisting of an integral expression only. The advantage of this version of the source function is that one may work out the integration with respect to the space coordinate first and apply the residue lemma afterwards. In the case of a zero draft platform this approach resulted in the dispersion relation in the plate region and an algebraic set of equations for the coefficients of the deflection only. Here we derive a coupled algebraic set of equations for the expansion coefficients of the potential in the fluid region and the deflection.

Mathematical formulation for the finite draft problem

The fluid is ideal, so we introduce the velocity potential $\mathbf{V}(\mathbf{x},t) = \nabla\Phi(\mathbf{x},t)$, where $\mathbf{V}(\mathbf{x},t)$ is the fluid velocity vector. Hence $\Phi(\mathbf{x},t)$is a solution of the Laplace equation $\Delta\Phi = 0$ in the fluid, together with the linearized kinematic condition, $\Phi_z = \tilde{w}_t$, and dynamic condition, $p/\rho = -\Phi_t - g\tilde{w}$, at the mean water surface $z = 0$, where $\tilde{w}(x,y,t)$ denotes the free surface elevation, and ρ is the density of the water. The linearized free surface condition outside the platform, $z = 0$ and $(x,y) \in \mathcal{F}$, becomes:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0. \tag{1}$$

To describe the vertical deflection $\tilde{w}(x,y,t)$, we apply the isotropic thin-plate theory and use the kinematic and dynamic condition to arrive at the following equation for Φ at $z = -d$ in the platform area $(x,y) \in \mathcal{P}$:

$$\left\{\frac{D}{\rho g}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + \frac{m}{\rho g}\frac{\partial^2}{\partial t^2} + 1\right\}\frac{\partial\Phi}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi}{\partial t^2} = 0. \tag{2}$$

We assume that the velocity potential is a time-harmonic wave function, $\Phi(\mathbf{x},t) = \phi(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}$. We introduce the following parameters: $K = \frac{\omega^2}{g}$, $\mu = \frac{m\omega^2}{\rho g}$, $\mathcal{D} = \frac{D}{\rho g}$. In a practical situation the total length l of the platform is a few thousand meters. The potential of the undisturbed incident wave is given by:

$$\phi^{\mathrm{inc}}(\mathbf{x}) = \frac{g\zeta_\infty}{\mathrm{i}\omega}\frac{\cosh(k_0(z+h))}{\cosh(k_0 h)}\exp\{\mathrm{i}k_0(x\cos\beta + y\sin\beta)\} \tag{3}$$

where ζ_∞ is the wave height in the original coordinate system, ω the frequency, while the wave number k_0 is the positive real solution of the dispersion relation,

$$k_0\tanh(k_0 h) = K, \tag{4}$$

for finite water depth. We restrict ourselves to the case of normal incidence, $\beta = 0$.

In the two-dimensional case, (x,z)-plane, the expression for the total potential becomes:

$$\begin{aligned}2\pi\phi(x,z) = 2\pi\phi^{\mathrm{inc}}(x,z) &+ \int_{-d}^{0}\left(\phi(0,\zeta)\frac{\partial\mathcal{G}(x,z;0,\zeta)}{\partial\xi} - \phi(l,\zeta)\frac{\partial\mathcal{G}(x,z;l,\zeta)}{\partial\xi}\right)\mathrm{d}\zeta \\ &+ \int_{0}^{l}\left(\phi(\xi,-d)\frac{\partial\mathcal{G}(x,z;\xi,-d)}{\partial\zeta} - \frac{\partial\phi(\xi,-d)}{\partial\zeta}\mathcal{G}(x,z;\xi,-d)\right)\mathrm{d}\xi.\end{aligned} \tag{5}$$

We continue with the two-dimensional case. The Green's function $G(x,z;\xi,\zeta)$ for the two dimensional case can be derived by means of a Fourier transform with respect to the x-coordinate. It has the form:

$$G(x,z;\xi,\zeta) = \int_{-\infty}^{\infty}\frac{1}{\gamma}\frac{K\sinh\gamma z + \gamma\cosh\gamma z}{K\cosh\gamma h - \gamma\sinh\gamma h}\cosh\gamma(\zeta+h)\,\mathrm{e}^{\mathrm{i}\gamma(x-\xi)}\,\mathrm{d}\gamma \quad \text{for } z > \zeta \tag{6}$$

and

$$G(x,z;\xi,\zeta) = \int_{-\infty}^{\infty}\frac{1}{\gamma}\frac{K\sinh\gamma\zeta + \gamma\cosh\gamma\zeta}{K\cosh\gamma h - \gamma\sinh\gamma h}\cosh\gamma(z+h)\,\mathrm{e}^{\mathrm{i}\gamma(x-\xi)}\,\mathrm{d}\gamma \quad \text{for } z < \zeta. \tag{7}$$

Semi analytic solution

In the formulation we restrict ourselves to the semi-infinite platform case. Results are also shown for the strip case. We eliminate in relation (5) the function $\phi(\xi,-d)$ by using equation (1) and replace $\phi_\zeta(\xi,-d)$ by $-\mathrm{i}\omega w(\xi)$. Thus we obtain,

$$\begin{aligned}2\pi\phi(x,z) &= 2\pi\phi^{\mathrm{inc}}(x,z) + \int_{-d}^{0}\phi(0,\zeta)\frac{\partial\mathcal{G}(x,z;0,\zeta)}{\partial\xi}\,\mathrm{d}\zeta \\ &-\mathrm{i}\omega\int_{0}^{\infty}\left(\frac{1}{K}\left(\mathcal{D}\frac{\partial^4}{\partial\xi^4} - \mu + 1\right)w(\xi)\frac{\partial\mathcal{G}(x,z;\xi,-d)}{\partial\zeta} - w(\xi)\,\mathcal{G}(x,z;\xi,-d)\right)\mathrm{d}\xi.\end{aligned} \tag{8}$$

We assume that the deflection $w(x)$ can be written as an expansion in exponential functions, truncated at $N+2$ terms of the form,

$$w(x) = \zeta_\infty\sum_{n=0}^{N+1} a_n\,\mathrm{e}^{\mathrm{i}\kappa_n x}. \tag{9}$$

If we consider κ_n's with either real positive values or, if they are complex , with positive imaginary part, then the first part of expression (9) expresses modes traveling and evanescent to the right. The second part then describes modes traveling and evanescent to the left.

Furthermore we expand the potential function for $x \leq 0$ and $x \geq l$ in series of orthogonal eigenfunctions, truncated at N terms

$$\phi(x,z) = \frac{g\zeta_\infty}{\mathrm{i}\omega}\left(\frac{\cosh k_0(z+h)}{\cosh k_0 h}\,\mathrm{e}^{\mathrm{i}k_0 x} + \sum_{n=0}^{N-1}\alpha_n \frac{\cosh k_n(z+h)}{\cosh k_n h}\,\mathrm{e}^{-\mathrm{i}k_n x}\right) \quad \text{for } x \leq 0 \quad (10)$$

where the k_n's are the positive real and positive imaginary roots of the dispersion relation (4). The difference in the number of expansion functions in (9) is due to the fact that we have two boundary conditions at the edge of the plate. The coefficient α_0 is the reflection coefficients. it should be noticed that the potential under the platform is **not** expanded in a set of orthogonal eigenfunctions. By the way, such a set does not exist. Extension of the solution along the bottom of the platform in the flow region is simply done by application of (8). We have introduced $2N+2$ unknown coefficients. Next we derive an algebraic set of equations for these coefficients. The values for κ_n follows from a 'dispersion' relation, yet to be determined.

First we take (x,z) at the bottom of the plate, this leads to the following equation

$$\begin{aligned} 2\pi\left(\mathcal{D}\frac{\partial^4}{\partial x^4} - \mu + 1\right) w(x) &= -2\pi\frac{K}{\mathrm{i}\omega}\phi^{\mathrm{inc}}(x,-d) - \frac{K}{\mathrm{i}\omega}\int_{-d}^{0}\phi(0,\zeta)\frac{\partial \mathcal{G}(x,-d;0,\zeta)}{\partial \xi}\,\mathrm{d}\zeta \\ &+ \lim_{z\uparrow -d}\int_0^\infty \left(\left(\mathcal{D}\frac{\partial^4}{\partial \xi^4} - \mu + 1\right) w(\xi)\frac{\partial \mathcal{G}(x,z;\xi,-d)}{\partial \zeta} - K w(\xi)\,\mathcal{G}(x,z;\xi,-d)\right)\mathrm{d}\xi. \end{aligned} \quad (11)$$

We take the limit in the last integral after we have carried out the spatial integrations analytically. This means that we keep the factor 2π in the left hand side of the equation. The commonly used factor π and principle value integral may be obtained by taking the limit first. However, it is more convenient to avoid the principle value integral in our approach. In the first integral on the right-hand side we insert for the Green function the series expansion and for the potential function the expansion (10), while in the second integral we use (7) for the Green function and (9) for the deflection. In the first integral integration with respect to ζ and in the last integral the integration with respect to ξ can be carried out. Next we close the remaining contour of integration in the complex γ-plane.

If we now equalize the coefficients of $\mathrm{e}^{\mathrm{i}\kappa_n x}$, we obtain the following 'dispersion' relation for κ_n, the κ_n's are the zero's of

$$(\mathcal{D}\kappa^4 - \mu - 1)\kappa\tanh\kappa(h-d) = K$$

At the bottom we compare the remaining exponential terms and at the frontend the hyperbolic cosine terms. We obtain for $i = 0,\cdots,N-1$:

$$\begin{aligned} &\sum_{n=0}^{N-1}\frac{\alpha_n}{\cosh k_n h}\mathcal{K}_{i,n} - \sum_{n=0}^{N+1}\frac{a_n}{\kappa_n - k_i}\left((\mathcal{D}\kappa_n^4 - \mu + 1)\sinh k_i(h-d) - \frac{K}{k_i}\cosh k_i(h-d)\right) \\ &= \delta_i^0\frac{hk_0^2 - hK^2 + K}{(k_0^2 - K^2)\cosh k_0 h)} - \frac{\mathcal{K}_{i,0}}{\cosh k_0 h}, \end{aligned}$$

At the frontend we obtain for $i = 0, \cdots, N-1$:

$$\frac{hk_i^2 - hK^2 + K}{(k_i^2 - K^2)\cosh k_i h}\alpha_i - \sum_{n=0}^{N-1} \frac{\alpha_n}{\cosh k_n h}\mathcal{K}_{i,n}$$
$$- \sum_{n=0}^{N+1} \frac{a_n}{\kappa_n + k_i}\left((\mathcal{D}\kappa_n^4 - \mu + 1)\sinh k_i(h-d) - \frac{K}{k_i}\cosh k_i(h-d) \right) = \frac{1}{\cosh k_0 h}\mathcal{K}_{i,0}$$

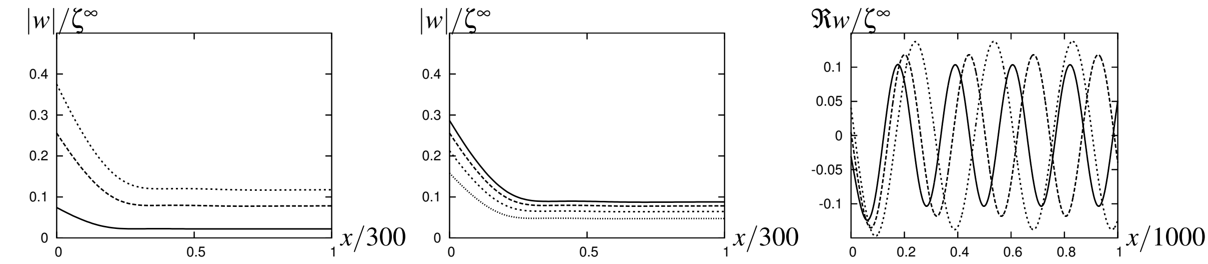
where the coefficients $\mathcal{K}_{i,n}$ are defined as:

$$2\,\mathcal{K}_{i,n} = \frac{k_i + k_n}{\sinh(k_i + k_n)h - \sinh(k_i + k_n)(h-d)} + \frac{k_i - k_n}{\sinh(k_i - k_n)h - \sinh(k_i - k_n)(h-d)}.$$

Together with the two edge conditions we have $2N+2$ equations for the $2N+2$ coefficients.

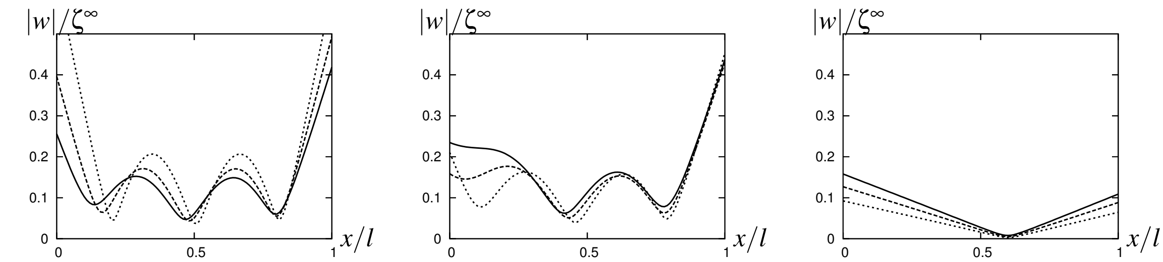
Results

We show some results for the half-plane and the strip.



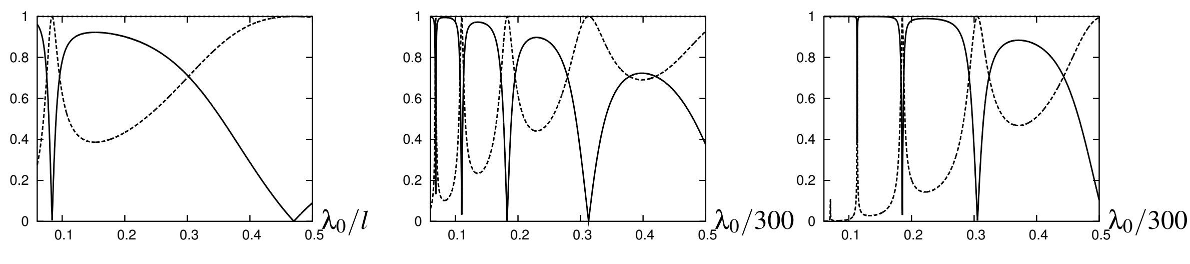
(a) $d = 2$ m, $\lambda = 150,\ 90,\ 30$ m (b) $d = 0,\ 2,\ 4,\ 6$ m, $\lambda = 90$ m (c) $h = 100,\ 20,\ 10$ m, $\lambda_0 = 100$ m

Figure 1: $\mathcal{D} = 10^7$ m^4, $h = 10$ m in (a) and (b), $d = 5$ m in (c)



(a) $\mathcal{D} = 10^7$ m^4, $\lambda/l = 0.3$ (b) $\mathcal{D} = 10^7$ m^4, $\lambda/l = 0.5$ (c) $\mathcal{D} = 10^{10}$ m^4, $\lambda/l = 0.5$

Figure 2: $l = 300$ m, $d = 0, —, 2, ——, 4, \cdots$ m



(a) $d = 0$ m and $l = 300$ m (b) $d = 2$ m and $l = 1000$ m (c) $d = 8$ m, and $l = 1000$ m

Figure 3: Reflection and transmission coefficients for $\mathcal{D} = 10^7$ m^4, $h = 100$ m