HYDRODYNAMIC LOADS ON FLAT PLATE ENTERING WATER

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SUMMARY

We consider a two-dimensional problem of floating plate which starts suddenly to penetrate water. The analysis is focused on early stage, during which the hydrodynamic loads are high. The liquid is assumed ideal and incompressible, gravity and surface tension effects are not taken into account. Method of matched asymptotic expansions is used to derive second-order uniformly valid solution of the problem. Non-dimensional plate displacement plays the role of the small parameter of the problem. The initial flow close to the plate edges is approximately self-similar and is governed by non-linear boundary-value problem with unknown shape of the free surface. The non-linear self-similar inner solution is matched to the second-order outer solution and is obtained numerically by the boundary-element method. The pressure distribution along the plate is obtained with the help of the nonlinear Bernoulli equation. The hydrodynamic pressure is integrated asymptotically with the aim to derive evolution of the hydrodynamic force acting on the plate during the early stage of impact. It is shown that the inner solution provides important contribution to the hydrodynamic force. We obtained that the initial asymptotics of the loads involve negative non-integer powers of the plate displacement and the log-term.

1. INTRODUCTION

The initial stage of two-dimensional flow caused by a sudden vertical motion of a plate initially floating on a still liquid surface is considered. No air is assumed entrained at the plate-liquid interface. The liquid flow is assumed plane, symmetric and potential. Both surface tension and the gravity are not taken into account. Viscous effects are assumed of minor importance and are neglected. The liquid is assumed ideal and incompressible. We shall obtain initial asymptotics of the hydrodynamic force acting on the moving plate.

The two-dimensional problem, which is under consideration in the present study, has rather limited application. However, the corresponding axisymmetric problem of circular disk impact (see [1]) is of importance in the processes of water entry of a body with flat nose. We expect that the asymptotic analysis developed in this paper for two-dimensional case will be helpful for analysis of three-dimensional problems of water impact.

It is well known (see [1], [2], [3]) that independently of the model adopted to describe the flow and loads, due to initial mismatch of the boundary conditions at the water line, an initial time interval exists during which hydrodynamic loads provided by numerical simulations are meaningless. This is why it is suggested to obtain the initial asymptotics of the flow and loads and use them as initial data for long-time solvers.

Such initial asymptotics were obtained for 2D problem of a floating flared body, which starts suddenly to enter water (see [4]). The flow domain was divided into the main flow region, where second-order outer velocity potential has been obtained, and small vicinities of the intersection points, where the leading order inner solution was computed and matched to the outer solution. The dimension of the vicinities was dependent on the penetration depth and the deadrise angle of the floating body at the intersection points. Uniformly valid pressure distribution was also obtained and used to derive the initial asypmtotics of the hydrodynamic force acting on the body during initial stage of its motion. It was obtained that non-integer powers of body displacement appear in the initial asymptotics of the loads. For bodies with the flare angle less than 45^o, negative powers were discovered in the initial asymptotics. Numerical calculations in [4] were performed for the case of floating wedge impact. However, the developed theory can be used also for the problem of flat plate impact (see section 2). It should be noticed that for a flat plate the second-order outer pressure distribution is not integrable, in contrast to the case of floating wedge impact, which requires more careful matching procedure and special care in deriving the force asymptotics valid for small times (see section 4). It is shown in this paper that the initial asymptotics of the loads acting on a flat plate involve not only negative powers of the plate displacement but also log-term.

The problem of flat plate impact was analyzed both numerically and theoretically in [5] for constant velocity of the plate after the impact. The inner solution, which is valid close to the plate edges, has been obtained and matched to the leading order outer solution in the main flow region. Note that the first-order outer solution was given by the pressure-impulse theory (see [5]). The asymptotic analysis presented in [5] is generalized in section 3 to the case of plate entry at variable velocity. The results from [4] are taken into account in section 3.

The asymptotic analysis of initial stage of plate impact is performed by using non-dimensional variables. Half-length of the plate L is taken as the length scale and the initial plate velocity V_0 as the velocity scale. The time scale is equal to L/V_0, the pressure scale $\rho_0 V_0^2$ and the scale of the hydrodynamic force is $\rho_0 V_0^2 L$ according to [4], where ρ_0 is the liquid density. The non-dimensional penetration depth $h(t)$ is positive with $h(0) = 0$ and $\dot{h}(0) = 1$, where an overdot stands for the time derivative.

2. SECOND-ORDER OUTER SOLUTION

The irrotational flow caused by plate impact is described by the complex velocity potential $w(z, t) = \varphi(x, y, t) + i\psi(x, y, t)$, where $z = x + iy$, $\varphi(x, y, t)$ is the velocity potential and

$\psi(x,y,t)$ is the stream function. The complex potential $w(z,t)$ is an analytical function in the flow region $\Omega(t)$, decays at the infinity, $x^2+y^2\to\infty$, and satisfies the following boundary conditions

$$\psi=\dot{h}(t)x \quad (y=-h(t),|x|<1), \tag{1}$$

$$\varphi_t+\frac{1}{2}|\nabla\varphi|^2=0, \varphi_y=\eta_x\varphi_x+\eta_t \quad (y=\eta(x,t)), \tag{2}$$

where equation $y=\eta(x,t)$ describes the elevation of the liquid free surface, the function $\eta(x,t)$ can be multi-valued, $\eta(-x,t)=\eta(x,t)$, $\eta(x,0)=0$ where $|x|>1$ and $\eta(x,t)\to 0$ as $|x|\to\infty$. Once the velocity potential $\varphi(x,y,t)$ has been obtained, the non-dimensional hydrodynamic pressure $p(x,y,t)$ is calculated as

$$p(x,y,t)=-\varphi_t-\frac{1}{2}|\nabla\varphi|^2. \tag{3}$$

The vertical force $F(t)$ acting on the moving plate is given as

$$F(t)=\int_{-1}^{1}p[x,-h(t),t]\,\mathrm{d}x. \tag{4}$$

We shall determine the asymptotic behaviors of both the complex potential $w(z,t)$, $z=x+iy$, and the hydrodynamic force $F(t)$ as $t\to 0$ up to the terms of the orders $o(t)$ and $o(1)$, respectively. Here $o(1)$ designates the terms which tend to zero and $o(t)$ the terms which tend to zero faster than t as $t\to 0$.

The theory developed in [4] provides that the second-order complex velocity potential is given as

$$w(z,t)=\dot{h}(t)w_0(z)+h\dot{h}w_1(z)+o(t), \tag{5}$$

where the first term corresponds to the pressure-impulse solution and the second term describes the evolution of the flow after the impact. In the case of plate impact we find

$$w_0(z)=i\Big(z-\sqrt{z^2-1}\Big),$$

$$w_1(z)=\frac{1}{2}[w_0'(z)]^2+\frac{iC_0(t)}{\sqrt{z^2-1}},$$

where $C_0(t)$ is a real function which should be determined by using the matching between the second-order outer and the inner solution. The velocity potential along the plate, $|x|<1$, $y=0$, is obtained in the form

$$\varphi=-\dot{h}\sqrt{1-x^2}+\frac{h\dot{h}/2}{1-x^2}-h\dot{h}-\frac{h\dot{h}C_0(t)}{\sqrt{1-x^2}}+o(t). \tag{6}$$

Equation (6) indicates that the outer solution is not valid close to the plate edges, where the second-order term becomes of the same order of magnitude as the leading order term. Dimension of the vicinity of the plate edge, where the outer asymptotics (6) is not valid, can be estimated by equating the orders of the first and the second terms in (6), which is rewritten in the inner variables u and v

$$x=1+a(t)u, \quad y=a(t)v.$$

Along the plate, we have $u=-r$, where r is the distance from the plate edge in the inner variables, and equation (6) takes the form

$$\varphi=-\dot{h}\sqrt{ar}\sqrt{2+ar}+\frac{h\dot{h}/2}{ar(2+ar)}$$
$$-h\dot{h}-\frac{h\dot{h}C_0(t)}{\sqrt{ar}\sqrt{2+ar}}+o(t), \tag{7}$$

where the first and the second terms are of the same order as $h\to 0$ if and only if

$$a(t)=[Bh(t)]^{\frac{2}{3}}.$$

The positive constant B is not defined in this analysis but the results of [5] show that it is convenient to assign $B=3/\sqrt{2}$. With this value of the constant B equation (7) can be presented as

$$\varphi=\sqrt{2}a^{\frac{1}{2}}\dot{h}\Big\{-\sqrt{r}+\frac{1}{12r}-\frac{hC_0(t)}{2a}\frac{1}{\sqrt{r}}\Big\}+O(h^{\frac{2}{3}}). \tag{8}$$

Note that we keep the term with C_0 in (8) because we do not know the order of this coefficient as $h\to 0$. In order to determine this coefficient and to resolve the singularity of the velocity potential close to the plate edge, we need to obtain the inner solution of the plate impact problem.

3. LEADING ORDER INNER SOLUTION

The inner variables are introduced as

$$x=1+a(t)u, \quad y=a(t)v,$$
$$\varphi=\sqrt{2}a^{\frac{1}{2}}\dot{h}\phi(u,v,t), \quad \eta=a\zeta(u,t). \tag{9}$$

By substituting (9) into boundary conditions (1), (2), enforcing the Kutta conditions at the plate edge and by using equalities $a\dot{a}=\sqrt{2a}\dot{h}$, $2a/\dot{a}=3h/\dot{h}$, we obtain the boundary-value problem, which governs the non-linear flow near the plate edge

$$\phi_{uu}+\phi_{vv}=0 \quad \text{in } \Omega \tag{10}$$

$$\phi_v=-\sqrt{a/2} \quad (v=0,u<0), \tag{11}$$

$$\phi-2(u\phi_u+v\phi_v)+\phi_u^2+\phi_v^2=-(3h/\dot{h})[\phi_t+\ddot{h}\phi], \tag{12}$$

$$\zeta-(\zeta_u u+\zeta_v v)+\phi_u\zeta_u-\phi_v=-\frac{3h}{2\dot{h}}\zeta_t \qquad (v=\zeta(u,t)), \tag{13}$$

$$\zeta(0,t)=-\frac{h}{a}, \qquad \zeta_u(0,t)=0, \tag{14}$$

$$\phi\to\sqrt{r}\sin\theta/2 \quad \text{as } r=\sqrt{u^2+v^2}\to\infty\ , \tag{15}$$

where θ is the angular coordinate, $u=r\cos\theta$ and $v=r\sin\theta$. Equation (15) is obtained by enforcing the matching between the outer limit of the inner solution and the inner limit of the outer solution given by equation (5). The Kutta conditions (14) imply that we are searching for the free surface shape, which is attached tangentially to the plate edges at any time.

In the leading order as $h\to 0$, the right-hand sides of equations (11) - (14) should be taken zero. Therefore, in the leading order the inner flow is self-similar, non-linear and with unknown in advance shape of the free surface. The corresponding boundary-value problem is identical to that derived in [5] for the case of plate motion at a constant velocity. The solution of this problem was obtained numerically in [5]. Asymptotic analysis provides that the inner velocity potential along the plate in the far field behaves as

$$\phi = -\sqrt{r} + \frac{C_1}{\sqrt{r}} + \frac{1}{12r} + o(r^{-1}), \tag{16}$$

when $r \to \infty$. The coefficient C_1 in the far-field asymptotics (16) was evaluated as a part of the inner solution. It was found that $C_1 \approx -0.41$. Comparing the inner limit of the second-order outer solution (8) with the far-field asymptotics of the leading order inner solution (16), we obtain that these solutions match each other if

$$C_0(t) = 2a(t)|C_1|/h. \tag{17}$$

4. PRESSURE DISTRIBUTION

Equation (17) makes it possible to present the second-order velocity potential along the plate as

$$\varphi = -\dot{h}\sqrt{1-x^2} + \frac{h\dot{h}/2}{1-x^2} - \frac{h^{\frac{2}{3}}\dot{h}\tilde{C}_0}{\sqrt{1-x^2}} - h\dot{h} + o(t), \tag{18}$$

where $\tilde{C}_0 = 2|C_1|B^{\frac{2}{3}}$. The pressure in the main flow region is calculated with the help of equation (3), where we disregard terms which tend to zero as $h \to 0$. By using the body boundary condition, we find

$$p(x,0,t) = -\varphi_t(x,0,t) - \frac{1}{2}\varphi_x^2(x,0,t) - \frac{1}{2}\dot{h}^2 + o(1). \tag{19}$$

Substituting (18) into (19) and omitting the term of the order of $o(1)$, we obtain

$$p(x,0,t) = \ddot{h}\sqrt{1-x^2} + \dot{h}^2 - \frac{\dot{h}^2}{1-x^2} +$$

$$+\frac{2}{3}\tilde{C}_0\dot{h}^2 h^{-\frac{1}{3}}\frac{1}{\sqrt{1-x^2}} + o(1). \tag{20}$$

Note that the third term in (20) is not integrable.

In the inner region, the pressure is given as

$$p(u,0,t) = -\frac{\dot{h}^2}{a}[\phi_u^2 + \phi - 2u\phi_u + o(1)]. \tag{21}$$

Equation (6) provides for $v = 0$ and $u = -r$

$$\phi_u^2 + \phi - 2u\phi_u = \phi_r^2 + \phi - 2r\phi_r = \frac{2C_1}{\sqrt{r}} + \frac{1}{2r} + P(r),$$

where $rP(r) \to 0$ as $r \to \infty$. It can be shown that the pressure in the inner region can be presented in the form

$$p(u,0,t) = -\frac{\dot{h}^2}{a}\Big[\frac{\alpha}{\sqrt{r}+\beta} + S(r) + o(1)\Big], \tag{22}$$

where $\alpha = 2C_1$, $\beta = (4|C_1|)^{-1}$ and $S(r)$ is an integrable function on the interval $0 < r < \infty$. In numerical calculations we compute the potential distribution $\phi(r,0)$ along the plate, $r > 0$, together with its first derivative $\phi_r(r,0)$ and then evaluate the function $S(r)$ with the formula

$$S(r) = \phi_r^2 + \phi - 2r\phi_r - \frac{\alpha}{\sqrt{r}+\beta}.$$

5. HYDRODYNAMIC FORCE

It is seen that both the outer (20) and the inner (22) pressure distributions are not integrable. In order to evaluate the total hydrodynamic force by using equation (4), we divide the plate surface into two regions: the main region, where $0 < x < 1 - a\lambda$, and the inner one, where $1 - a\lambda < x < 1$. Here λ is a large parameter such that $\lambda \gg 1$ but $a\lambda^{\frac{3}{2}} \ll 1$. For example, one can take $\lambda = a^{-\frac{1}{2}}$. Note that time t and the function $a(t)$ are considered as parameters now. In the main region, the pressure is given by equation (20) and in the inner region by equation (22). It is suggested to calculate separately the contributions of the pressure in the main region and in the inner region to the total hydrodynamic force and to check that the total force is independent of the parameter λ up to the terms of the order of $o(1)$ as $h \to 0$. This technique has been used in [6] for calculations of the second-order hydrodynamic force within the higher-order Wagner theory of wave impact.

The total force acting on the plate is calculated as

$$F(t) = \lim[F_{main}(t) + F_{inner}(t)] + o(1), \tag{23}$$

where

$$F_{main}(t) = 2\int_0^{1-a\lambda} p(0,-h,t)\mathrm{d}x, \tag{24}$$

$$F_{inner}(t) = 2\int_{1-a\lambda}^{1} p(0,-h,t)\mathrm{d}x. \tag{25}$$

The limit in (23) is taken as $a \to 0$, $\lambda \to \infty$ and $a\lambda^{\frac{3}{2}} \to 0$ at the same time.

We start with equation (25), where the pressure is given by (22). By using $x = 1 - ar$ and (22), we obtain

$$F_{inner}(t) = -2\dot{h}^2\int_0^{\lambda}\Big[\frac{\alpha}{\sqrt{r}+\beta} + S(r) + o(1)\Big]\mathrm{d}r.$$

Neglecting the terms, which give small contribution as $\lambda \to \infty$ and $a \to 0$, one obtains

$$F_{inner}(t) = 4|C_1|\dot{h}^2\int_0^{\lambda}\frac{\mathrm{d}r}{\sqrt{r}+\beta} - 2\dot{h}^2 J + o(1), \tag{26}$$

where

$$J = \int_0^{\infty} S(r)\mathrm{d}r$$

and

$$\int_0^{\lambda}\frac{\mathrm{d}r}{\sqrt{r}+\beta} = 2\sqrt{\lambda} - \beta\ln\lambda + 2\beta\ln\beta + o(1). \tag{27}$$

In equation (24), the pressure is given by (20), which provides

$$F_{main}(t) = 2\ddot{h}\int_0^{1-a\lambda}\sqrt{1-x^2}\mathrm{d}x - 2\dot{h}^2\int_0^{1-a\lambda}\frac{\mathrm{d}x}{1-x^2}$$

$$+2\dot{h}^2[1-a\lambda]+\frac{4}{3}\tilde{C}_0\dot{h}^2h^{-\frac{1}{3}}\int_0^{1-a\lambda}\frac{\mathrm{d}x}{\sqrt{1-x^2}}+o(1). \quad (28)$$

Special care is required to obtain the asymptotics of the third integral in (28). We find

$$\int_0^{1-a\lambda}\frac{\mathrm{d}x}{\sqrt{1-x^2}}=\frac{\pi}{2}-2\sqrt{\frac{a\lambda}{2}}\int_0^1\frac{\mathrm{d}\xi}{\sqrt{1-\xi^2a\lambda/2}}$$

$$=\frac{\pi}{2}-\sqrt{2a\lambda}+O([a\lambda]^{\frac{3}{2}}).$$

By substituting the latter asymptotic formula in (28) and evaluating the integrals, we obtain

$$F_{main}(t)=\frac{\pi}{2}\ddot{h}-\dot{h}^2\ln\Big(\frac{2-a\lambda}{a\lambda}\Big)+2\dot{h}^2$$

$$+\frac{4}{3}\tilde{C}_0\dot{h}^2h^{-\frac{1}{3}}\Big[\frac{\pi}{2}-\sqrt{2a\lambda}+O([a\lambda]^{\frac{3}{2}})\Big]+o(1), \quad (29)$$

where the terms, which tend to zero as $a\to 0$, $\lambda\to\infty$ and $a\lambda^{\frac{3}{2}}\to 0$, as designated as $o(1)$. By algebra

$$\frac{4}{3}\tilde{C}_0\dot{h}^2h^{-\frac{1}{3}}\times\sqrt{2a\lambda}=8|C_1|\dot{h}^2\sqrt{\lambda}.$$

Equation (29) takes now the form

$$F_{main}(t)=\frac{\pi}{2}\ddot{h}+\dot{h}^2\Big[2-\ln 2+\frac{2}{3}\ln B\Big]$$

$$+\frac{2}{3}\dot{h}^2\ln h+\dot{h}^2\ln\lambda+\frac{2\pi}{3}\tilde{C}_0\dot{h}^2h^{-\frac{1}{3}}-8|C_1|\dot{h}^2\sqrt{\lambda}+o(1). \quad (30)$$

Substituting equations (26), (27) and (30) into (23), we note that the terms with λ cancel each other with the result

$$F(t)=\frac{\pi}{2}\ddot{h}+\gamma\dot{h}^2h^{-\frac{1}{3}}+\frac{2}{3}\dot{h}^2\ln h$$

$$+\dot{h}^2\Big[2-2\ln(4|C_1|)-\ln 2+\frac{2}{3}\ln B-2J\Big]+o(1), \quad (31)$$

where $\gamma=\frac{4\pi}{3}|C_1|B^{\frac{2}{3}}$. In (31), the first term comes from the leading order approximation and other terms describe the higher-order effects.

6. FREE-FALLING PLATE

We consider a plate of mass M per unit length, which falls down onto water surface from a height H. Without account for the aerodynamic force acting on the plate before the impact, the plate velocity $V(-0)$ just before the plate contact with water is given as $V(-0)=\sqrt{2gH}$, where g is the gravity acceleration. During the very early stage the first term in (31) is of the major importance and we obtain the well-known formula for the velocity of the plate $V(+0)$ at the end of this stage as

$$V(+0)=\frac{V(-0)}{1+\nu}, \quad \nu=\frac{\pi}{2m}, \quad m=\frac{M}{\rho_0L^2}$$

This is $V_0=V(+0)$, which is taken as the velocity scale in the present analysis.

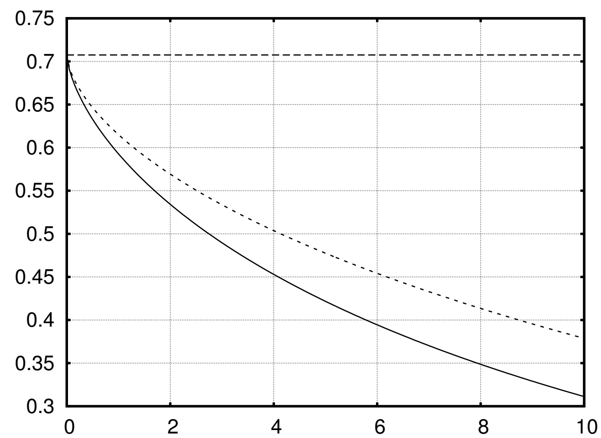
We perform here simplifies analysis of the plate motion after the impact by taking only two main terms in the hydrodynamic force (31). The plate motion after impact instant is governed in the non-dimensional variables by the equation

$$(m+\frac{\pi}{2})\ddot{h}\approx-\gamma\dot{h}^2h^{-\frac{1}{3}}.$$

The latter equation can be integrated with the initial conditions $h(0)=0$ and $\dot{h}(0)=1$ with the result

$$\dot{h}(t)\approx\exp[-\frac{3\gamma}{2m+\pi}h^{\frac{2}{3}}]. \quad (32)$$

Equation (32) shows that the plate velocity continues to decrease sharply after the impact instant due to the higher-order effects.



The plate velocity after the impact instant is shown in dimensional variables in figure 1 for the following impact conditions

$$M=50kg/m, \quad H=2m, \quad L=0.5m.$$

Calculations provide $V(-0)=6.26m/s$, $m=0.2$, $\gamma=2.83$ and $V_0=0.7m/s$. In figure 1, the solid line corresponds to the plate velocity provided by equation (32), the dashed line is the prediction by the leading order theory and the dotted line gives the plate velocity evolution, when three first terms are taken into account in equation (31). The vertical axis is for the plate velocity measured in meters per second and the horizontal axis is for the penetration depth measured in centimeters. It is seen that that the log-term in (31) provides important contribution. However, the most important finding is that the plate velocity sharply drops after the impact instant due to the higher-order terms in the initial asymptotics of the hydrodynamic force.

7. REFERENCES

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