Rapid acceleration of a free-surface pressure system in forward motion

Nikolay KUZNETSOV

Laboratory for Mathematical Modelling of Wave Phenomena,
Institute for Problems in Mechanical Engineering, Russian Academy of Sciences,
V.O., Bol'shoy pr. 61, St. Petersburg 199178, Russian Federation,
E-mail: nikuz@wave.ipme.ru

1 Introduction; statement of the problem

Our aim is to investigate the effect of a rapid acceleration on the resistance to the rectilinear forward motion of a surface pressure distribution. The distribution is assumed to start its non-uniform motion when the free surface is horizontal and water rests over a variable bottom topography and submerged bodies. A two-scale expansion of a velocity potential allows us to derive an explicit time dependence for the resistance during the interval of acceleration. Moreover, we find how the resistance depends on the bottom topography that varies in the direction of motion.

Let an inviscid, incompressible fluid of density ρ (*e.g.*, water) occupy an infinite domain W that is contained in a horizontal layer of constant, possibly infinite, depth $d \in (0, +\infty]$ (see Fig. 1, where a two-dimensional sketch of geometry is shown). Cartesian coordinates (x, y, z) are chosen so that $F = \{-\infty < x, z < +\infty,\ y = 0\}$ coincides with the mean free surface, and the y-axis is directed vertically upwards. Along with F, there may be a sea-bed B and a surface S, which is the union of the wetted boundaries of all submerged bounded bodies; B and S are rigid, sufficiently smooth surfaces and each of them may be empty.

Let the free-surface pressure distribution be given by a smooth function $\mathcal{P}(x, z)$ at the initial moment of time $t = 0$. Let $\mathcal{P}$ have a compact support; that is, $\mathcal{P}$ vanishes outside a bounded two-dimensional region having the diameter D, and $\mathcal{P} \neq 0$ everywhere inside it. In what follows, it is convenient to apply dimensionless variables using the same notation for the variables and functions already introduced. We take D as the characteristic length, $(D/g)^{1/2}$ as the characteristic time interval, and $\rho D g$ as the characteristic pressure, where g is the acceleration due to gravity. The characteristic scales for other functions are generated by these three. We assume that the time dependence of the pressure distribution is as follows:

$$p(x, z; t) = \mathcal{P}\left(x - \int_0^t \mathcal{V}(\mu)\,\mathrm{d}\mu,\ z\right) \quad \text{for} \quad t \geq 0, \tag{1}$$

where $\mathcal{V}(t) \geq 0$ is the forward velocity. The latter is supposed to be a continuous function of $t \geq 0$, vanishing at $t = 0$ and depending also on a small parameter ϵ in the following way: $\mathcal{V}(t) = v(t/\epsilon) \geq 0$ and $v(\mu) \to V = \text{const} > 0$ as $\mu \to \infty$. Moreover, $Q(\mu) = V - v(\mu)$ must decay at infinity so that $\mu^m Q(\mu) \to 0$ as $\mu \to \infty$ for any $m = 1, 2, \ldots$

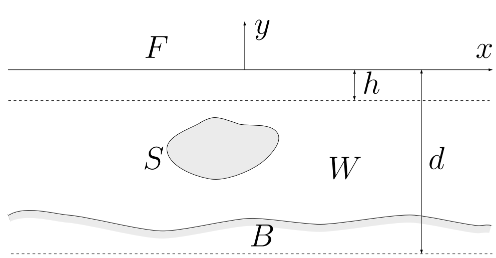

Figure 1: A two-dimensional definition sketch of the geometry.

The linearized theory of the irrotational unsteady water waves caused by the moving pressure distribution is formulated in terms of a velocity potential $\phi(X;t,\epsilon)$, $X=(x,y,z)$, (see, *e.g.*, [1]). It is natural to assume that ϕ belongs to the class of functions having finite the kinetic and potential energy:

$$\int_W |\nabla\phi|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_F \eta^2\,\mathrm{d}x\mathrm{d}z < \infty. \tag{2}$$

Here $\nabla=(\partial_x,\partial_y,\partial_z)$ and η denotes the free-surface elevation linked to ϕ and p by the linearized kinematic condition on the free surface:

$$\eta(x,z;t) = -\left[\partial_t\phi(x,0,z;t) + p(x,z;t)\right]. \tag{3}$$

For $t\ge 0$ the velocity potential ϕ must satisfy the following boundary value problem:

$$\nabla^2\phi = 0 \text{ in } W, \quad \partial_t^2\phi + \partial_y\phi = -\partial_t p \text{ on } F, \quad \partial_n\phi = 0 \text{ on } B\cup S, \tag{4}$$

where ∂_n indicates differentiation with respect to the unit normal directed into W. Equations (4) must be complemented by the following two initial conditions:

$$\phi(x,0,z;0) = 0, \quad \partial_t\phi(x,0,z;0) = -\mathcal{P}(x,z). \tag{5}$$

The meaning of the second condition (5) follows from (3) and (1) and expresses the fact that the free surface is horizontal at $t=0$. According to the first condition (5), equations (4), and condition (2), we have that $\phi(X;0,\epsilon)$ vanishes identically in W, which means that there is no initial motion of water.

Our aim is to construct an asymptotic expansion for ϕ valid as $\epsilon\to 0$. In order to understand what the assumption $\epsilon\ll 1$ means, one has to consider a velocity $v(\mu)$ that is equal to V identically for $\mu\ge 1$, in which case the velocity varies only during the initial time interval that is short in comparison with the characteristic time interval $(D/g)^{1/2}$.

2 Asymptotic expansion

The forward velocity $v(t/\epsilon)$ involves the so-called 'rapid' time $\tau=t/\epsilon$, and using this second time scale along with t, the right-hand-side term in the second equation (4) with p given by (1) can be written in the form:

$$-\partial_t p = v(\tau)\partial_x\mathcal{P}\left(x - Vt + \epsilon\alpha(\tau),\, z\right), \quad \text{where} \quad \alpha(\tau) = \int_0^\tau [V - v(\mu)]\,\mathrm{d}\mu. \tag{6}$$

In order to apply asymptotic procedure let us expand (6) into a sum of two series in powers of ϵ so that each series depends only on a single time scale τ or t. Thus we arrive at the following expansion for (6):

$$-\partial_t p = \sum_{m=0}^{\infty}\frac{\epsilon^m}{m!}\left\{\beta_m(\tau)\partial_x^{m+1}\mathcal{P}(x,z) + V[\alpha(\infty)]^m\partial_x^{m+1}\mathcal{P}(x-Vt,z)\right\},$$

where

$$\beta_m(\tau) = v(\tau)\left[\alpha(\tau) - V\tau\right]^m - V\left[\alpha(\infty) - V\tau\right]^m, \quad m = 0,1,\ldots$$

It is easy to check that

$$\beta_m(0) = -V[\alpha(\infty)]^m \quad \text{and} \quad \beta_m(\tau)\to 0 \quad \text{as} \quad \tau\to\infty.$$

Using a standard asymptotic procedure (see, *e.g.*, [1], ch. 10), one finds the velocity potential as the two-time scaled asymptotic series

$$\phi(X;t,\epsilon) = \sum_{m=0}^{\infty}\epsilon^m\left[\varphi_m(X;\tau) + \psi_m(X;t)\right]. \tag{7}$$

Here $\varphi_m(X;\tau)$ depends on τ explicitly, namely:
• $\varphi_0(X;\tau)$ and $\varphi_1(X;\tau)$ vanish identically for $0 \leq \tau < +\infty$ and $X \in \overline{W}$;

$$\bullet\ \varphi_m(X;\tau) = \sum_{k=1}^{[m/2]} u_{mk}(X) \frac{1}{(2k-1)!(m-2k)!} \int_\tau^\infty (\mu-\tau)^{2k-1} \beta_{m-2k}(\mu)\, \mathrm{d}\mu \text{ for } m = 2, 3, \dots,$$

where $[s]$ denotes the integer part of $s \in (-\infty, +\infty)$ and u_{mk} must be determined from the following boundary value problem:

$$\nabla^2 u_{mk} = 0 \quad \text{in} \quad W, \qquad \partial_n u_{mk} = 0 \quad \text{on} \quad B \cup S,$$
$$u_{mk} = \begin{cases} \partial_x^{m-1} \mathcal{P}(x,z) & \text{for} \quad k = 1, \\ -\partial_y u_{m-2,k-1} & \text{for} \quad k = 2, 3, \dots \end{cases} \quad \text{on} \quad F,$$

which is uniquely solvable under condition (2).

The functions ψ_m, $m = 0, 1, \dots$, must be determined from the following initial-boundary value problems:

$$\nabla^2 \psi_m = 0 \text{ in } W, \quad \partial_n \psi_m = 0 \text{ on } B \cup S, \quad \text{for } t \geq 0;$$
$$\partial_t^2 \psi_m + \partial_y \psi_m = V[\alpha(\infty)]^m \partial_x^{m+1} \mathcal{P}(x - Vt, z) \text{ on } F \quad \text{for } t \geq 0;$$
$$\psi_m(x,0,z;0) = -\varphi_m(x,0,z;0), \quad m = 0, 1, \dots;$$
$$\partial_t \psi_0(x,0,z;0) = -\mathcal{P}(x,z);$$
$$\partial_t \psi_m(x,0,z;0) = -\partial_\tau \varphi_{m+1}(x,0,z;0), \;\; m = 1, 2, \dots.$$

If water has constant depth and there are no submerged bodies, then it is possible to integrate explicitly the sequence of problems for φ_m and ψ_m. This can be performed in the same way as in [1], Section 10.1.3.

To justify the asymptotic formula (7) one has to estimate the remainder term

$$r_N(X;t,\epsilon) = \phi(X;t,\epsilon) - \sum_{m=0}^{N} \epsilon^m \left[\varphi_m(X;\tau) + \psi_m(X;t)\right].$$

This can be done in the same way as in [1], Chapter 10, and one arrives at the following result.

Let $\mathcal{P}$ belongs to the Sobolev space $H^{N+2}(F)$, then $\|r_N\|_{1/2} \leq C(N)\, \epsilon^{N+1}\, t\, \|\mathcal{P}\|_{N+2}$, which justifies the asymptotic formula (7)*. Here $\|\cdot\|_\ell$ denotes the norm in $H^\ell(F)$.*

3 Hydrodynamic corollaries

The asymptotic expansion (7) allows us to obtain two versions of asymptotic formulae for the following hydrodynamic characteristics: (i) the free-surface elevation; (ii) the horizontal component of the reaction of water to the forward motion of the pressure distribution, this component is referred to as the resistance. Formulae of the first type are valid on any finite subinterval of $t \geq 0$, whereas formulae of the second type are valid only during the period comparable with the acceleration time, that is, for $t = O(\epsilon)$.

The zero-order asymptotic formula for the free-surface elevation, valid on any finite subinterval of $t \geq 0$, is as follows:

$$\eta(x,z;t,\epsilon) = -\left[\partial_t \psi_0(x,0,z;t) + \mathcal{P}(x - Vt, z)\right] + O(\epsilon).$$

This formula means that up to a term $O(\epsilon)$ the free-surface elevation η is the same if either the pressure system instantly starts the forward motion at the limit speed V or the same system approaches the same speed during a time interval $O(\epsilon)$. The correction to the above asymptotics is equal to $-\epsilon\,[\partial_t\psi_1(x,0,z;t)+\alpha(\infty)\partial_x\mathcal{P}(x-Vt,z)]+O(\epsilon^2)$, where the contributions depending on the rapid time τ also cancel. This is natural in view of the assumption that the acceleration time scale ϵ is short in comparison with the gravitational time scale $(D/g)^{1/2}$.

Further asymptotic analysis of the free-surface elevation leads to the following formula:

$$\eta(x,z;t,\epsilon)=\frac{t^2}{2}\partial_y U(x,0,z)+O\left(\epsilon^3\right) \quad \text{for} \quad t=O(\epsilon). \tag{8}$$

Here U must be determined from the following boundary value problem:

$$\nabla^2 U=0 \text{ in } W, \quad \partial_n U=0 \text{ on } B\cup S, \quad U=\mathcal{P}(x,z) \text{ on } F. \tag{9}$$

Presumably, the similar dependence on time in (8) and in the falling body law is caused by the role of gravity in generating water waves. A consequence of formula (8) and the maximum principle for harmonic functions is that the point on the horizontal free surface, subjected to the maximum pressure at the initial moment, moves upwards after being released from the action of the pressure because of its forward motion.

Let us turn to the resistance which is equal to $R(t,\epsilon)=\int_{y=\eta(x,z;t)} p\,n_x\,\mathrm{d}\sigma$. Here n_x is the x-component of the unit normal to $y=\eta(x,z;t)$ and $\mathrm{d}\sigma$ denotes the element of surface area. Using (3) and the fact that p is given by (1), we obtain another representation:

$$R(t,\epsilon)=\int \partial_t\phi(x,0,z;t,\epsilon)\,\partial_x\mathcal{P}\left(x-\int_0^t v(\mu/\epsilon)\,\mathrm{d}\mu,\ z\right)\mathrm{d}x\mathrm{d}z, \tag{10}$$

where we integrate over the support of $\mathcal{P}$. The principal term in the asymptotics of $R(t,\epsilon)$ (it is valid on any finite time interval) arises when one changes ϕ and the integral over $(0,t)$ to ψ_0 and Vt, respectively, in (10).

For the initial interval, the asymptotics of resistance is given by the following formula:

$$R(t,\epsilon)=\frac{t^2}{4}\int_{B\cup S}|\nabla U|^2 n_x\,\mathrm{d}\sigma+O\left(\epsilon^3\right) \quad \text{for} \quad t=O(\epsilon). \tag{11}$$

An advantage of this formula is the dependence on the geometry of W through the solution U of a time-independent boundary value problem (9), which allows us to make qualitative conclusions for particular geometries.

Let $S=\emptyset$ and let B be a cylindrical surface having its generators parallel to the z-axis, that is, $B=\{-\infty<x,z<+\infty,\ y=-H(x)\}$, where $H(x)>0$. If H is monotone, then n_x has a fixed sign on B: $n_x\geq 0$ $(n_x\leq 0)$ when $-H$ decreases (increases). Therefore, *when the pressure distribution accelerates down (up) the bottom slope, the principal term in the asymptotic formula* (11) *is positive (negative).* If $S=\emptyset$ and B is horizontal, then the principal term in (11) vanishes. However, the third-order term can be found explicitly in this case and this term is negative. Thus during the rapid acceleration of the surface pressure the resistance can act in the direction opposite to the direction of motion as well as in the same direction. Earlier, a similar effect has been discovered in the paper [2] concerned with the problem of the wave-making resistance for a submerged body moving forward so that its velocity oscillates at a high frequency about a mean value.

References

[1] N. Kuznetsov, V. Maz'ya & B. Vainberg, *Linear Water Waves: A Mathematical Approach.* Cambridge University Press (2002) xvii+513 pp.

[2] N. Kuznetsov, Asymptotic analysis of wave resistance of a submerged body moving with oscillating velocity. *J. Ship Research* **37** (1993) 119–125.

Kuznetsov, N.
'Rapid acceleration of a free-surface pressure'

Discusser - T. Miloh:

Are your results valid only for 'rapid' acceleration? What happens if your pressure acceleration is 'moderate' or even slow?

You made a special point of the fact that the free-surface elevation is proportional at small-time to t? Since we know that both η and η_t are zero for $t \to 0$ is it not obvious that $\eta \sim t^2$ for $t \to 0$?

Reply:

There is a problem closely related to the problem of 'rapid' acceleration and this related problem is considered in detail in ch. 10 of 'Lineaer Water Waves: a Mathematical Approach' by N. Kuznetsov, V. Maz'ya, B. Vainberg, CUP 2002.

For that second problem numerical computations were produced for a particular case for which the explicity solution does exist. There is a good agreement between the exact and asymptotic solutions when ϵ is as large as 10.

Of course, the proportionality of η to t^2 is a rather simple fact, but the coefficient depending on (x, z) is also obtained in the presented approach.

Discusser - D.H. Peregrine:

Please explain how in the expression for ϕ_m at the top of p. 99 the $\int_\tau^\infty$ satisfies causality.

Reply:

The integral $\int_\tau^\infty = \int_\tau^1$ because $\alpha(\tau) \equiv \alpha(1)$ for $\tau \geq 1$ according to the definition of $\nu(\mu)$, and so $\beta_m(\tau) \equiv 0$ for $\tau \geq 1$ and $m = 0, 1, \ldots$. Hence the integral under consideration is NOT related to causality. The latter is buried in ψ_m, $m = 0, 1, \ldots$, that depends on the 'usual' time t.