A new approach to uniqueness for linear problems of wave–body interaction

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1. Introduction and statement of the problem

In the paper we shall consider the question of uniqueness in linear problems, which describe interaction between an ideal unbounded fluid and bodies located under the free surface of the fluid. In particular, it can be radiation of waves by forced motion of rigid bodies or diffraction of waves by fixed bodies. The problems appear within the framework of the surface wave theory under the assumptions that the motion is steady-state, irrotational and the oscillations have small amplitudes.

It is well-known (see, e.g. [3, $\S$ 2.2.1]) that uniqueness for the problems under consideration is equivalent to non-existence of trapped modes, i.e. unforced localized oscillations of fluid; corresponding velocity potential satisfies the homogeneous problem:

$$\Delta u = 0 \quad \text{in} \quad W, \tag{1}$$

$$\partial_y u - \nu u = 0 \quad \text{on} \quad F, \quad \partial_{\boldsymbol{n}} u = 0 \quad \text{on} \quad S, \tag{2}$$

$$\int_W |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}y + \nu \int_F |u|^2 \, \mathrm{d}\boldsymbol{x} < \infty, \tag{3}$$

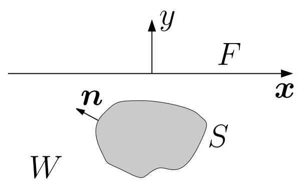

Fig. 1

where $\Delta = \partial_x^2 + \partial_y^2$ and $\boldsymbol{x} = x$ in the two-dimensional case, $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_y^2$ and $\boldsymbol{x} = (x_1, x_2)$ in the three-dimensional case; the origin of coordinate system is situated in the free surface and the fluid in absence of bodies occupies $y \leqslant 0$. Besides, W denotes the domain occupied by fluid, S is the surface of bodies and F is the free surface (see fig. 1). We shall consider the case when the fluid has infinite depth, but main results of the paper are applicable to the finite depth case too.

It is notable that for the seemingly simple problem very few criteria of uniqueness are known despite the long history of surface-wave theory. A good review of existing results can be found in the book [3]. The most universal approach is the method of integral equations: for bodies of arbitrary shape it guarantees unique solvability of the problem for all values of ν except for a finite (possibly empty) set of values, which cannot be defined within the scheme. Other approaches (e.g. the criterion of uniqueness [6]) guarantee absence of eigenvalues ν of the spectral problem (1)–(3) in some intervals under some restrictions on geometry. We emphasize that existence of trapped modes for some special classes of bodies has been shown numerically (see [4, 7]).

In the paper a new criterion of uniqueness in two- and three-dimensional problems is suggested. The method is universal: it is applicable for submerged bodies of arbitrary shape and without restrictions on the parameter ν. The criterion delivers effective procedure for numerical verification of uniqueness property, so that uniqueness can be established numerically for all values of ν except vicinities of isolated points, corresponding to trapped modes. The size of the vicinities is defined within the method and it decays to zero as step of discretization in the suggested approximation scheme decays to zero.

2. Criterion of uniqueness

We start with the results proved in [3, § 2.1]. Taking into account that the integral equations considered there are Fredholm ones, we can formulate the result of [3, § 2.1.2.3] as follows: both in two- and three-dimensional cases the problem (1)–(3) is uniquely solvable if and only if the homogeneous equations

$$-\mu(z) + T\mu(z) = 0, \quad -u(z) + T^{*}u(z) = 0, \quad z = (\boldsymbol{x}, y) \in S, \tag{4}$$

have only the trivial solution. Here

$$(T\mu)(z) = 2\int_S \mu(\zeta)\,\partial_{n(z)}G(z;\zeta)\,\mathrm{d}s_\zeta, \quad (T^{*}u)(z) = 2\int_S u(\zeta)\overline{\partial_{n(\zeta)}G(z;\zeta)}\,\mathrm{d}s_\zeta, \quad \zeta = (\boldsymbol{\xi}, \eta),$$

and G is the Green function of the problem satisfying equation $\Delta G = -\,\delta_\zeta(z)$ in the fluid.

It is possible to formulate a uniqueness criterion in the form $\min|\alpha_i - 1| > \delta$, where α_i are eigenvalues of the operator T ($T\mu = \alpha_i\mu$). However, inconvenience of this idea is that the operator T is not self-adjoint, which creates essential difficulty in estimating proximity of its eigenvalues to their numerical approximation.

However, it turns out that a self-adjoint formulation for uniqueness criterion can be found. Since both equations (4) either have non-trivial solutions or not, action of operator $I - T^{*}$ from the left to the first equation (4) does not create new solutions. Thus, it can be shown that the problem (1)–(3) has a non-trivial solution if and only if a non-trivial solution exists to the equation

$$-\mu + \mathscr{T}\mu = 0, \quad \mathscr{T} = T + T^{*} - T^{*}T.$$

The operator $\mathscr{T}$ is self-adjoint. Besides, $\langle (I - T^{*})(I - T)v, v\rangle = \langle (I - T)v, (I - T)v\rangle \geqslant 0$ (here $\langle \bullet, \bullet\rangle$ is the scalar product in $L_2(S)$). Hence, $\langle \mathscr{T}v, v\rangle \leqslant \langle v, v\rangle$ and all eigenvalues λ_i of the operator $\mathscr{T}$ are submitted to the inequality $\lambda_i \leqslant 1$. Thus, $\lambda_1 = 1$ indicates existence of trapped modes and absence of non-trivial solutions to the problem (1)–(3) is equivalent to the condition

$$\lambda_1 < 1, \quad \text{where} \quad \lambda_1 = \max\{\lambda_i\}.$$

For practical application of the criterion an approximation $\hat{\mathscr{T}}$ of the operator $\mathscr{T}$ should be used. Let $\hat{T}$ be an approximation of the operator T, then it is convenient to define $\hat{\mathscr{T}} = \hat{T} + \hat{T}^{*} - \hat{T}^{*}\hat{T}$. It is easy to find that $\mathscr{T} - \hat{\mathscr{T}} = \epsilon + \epsilon^{*} - \epsilon^{*}\epsilon - \hat{T}^{*}\epsilon - \epsilon^{*}\hat{T} = \epsilon + \epsilon^{*} + \epsilon^{*}\epsilon - T^{*}\epsilon - \epsilon^{*}T$, where $\epsilon = T - \hat{T}$. Hence,

$$\delta = \|\mathscr{T} - \hat{\mathscr{T}}; L_2(S)\| \leqslant 2\|\epsilon; L_2(S)\|\left(1 + \min\left\{\|T; L_2(S)\|, \|\hat{T}; L_2(S)\|\right\}\right) + \|\epsilon; L_2(S)\|^2. \tag{5}$$

We denote $\hat{\lambda}_1 = \max\{\hat{\lambda}_i\}$, where $\hat{\lambda}_i$ are eigenvalues of $\hat{\mathscr{T}}$ (obviously, $\hat{\lambda}_i \leqslant 1$). Since $\mathscr{T}$ and $\hat{\mathscr{T}}$ are self-adjoint, theorem 4.10 [2] guarantees that $\hat{\lambda}_1 - \lambda_1 \leqslant \delta$. Finally, we can formulate sufficient condition of uniqueness in the form

$$\hat{\lambda}_1 + \delta < 1. \tag{6}$$

3. Approximation and numerical results

We split the surface S into N parts γ_i and consider the following approximation of the operator T

$$(\hat{T}\mu)(z) = \int_S \mu(\zeta)\,\hat{K}(z,\zeta)\,\mathrm{d}s_\zeta,$$

where $\hat{K}(z,\zeta) = \sum_{i,j=1}^N \chi_j(z)\,\chi_i(\zeta)\,|\gamma_j|^{-1}\int_{\gamma_j} K(z,\zeta_i)\,\mathrm{d}s_z$; $K(z,\zeta) = 2\,\partial_{n(z)}G(z,\zeta)$; $\{\zeta_i\}_1^N$ is a set of points, such that $\zeta_i \in \gamma_i$; $\chi_i(z) = 0$ when $z \notin \gamma_i$ and $\chi_i(z) = 1$ when $z \in \gamma_i$. The approximation is especially convenient in the two-dimensional case when $\int_{\gamma_j} K(z,\zeta_i)\,\mathrm{d}s_z = H(z_j',\zeta_i) - H(z_j'',\zeta_i)$, where $H(z,\zeta)$ is a complex conjugate to $G(z,\zeta)$ in z and z_j', z_j'' are the end-points of γ_j.

The operator $\hat{T}$ can be represented by a matrix. On computing the matrix for a given value of ν, we can find $\hat{\mathscr{T}}$, $\hat{\lambda}_1$ and $\|\hat{T}; L_2(S)\|$. Besides, obviously,

$$\big\|T - \hat{T}; L_2(S)\big\|^2 \leqslant \int_S\int_S \big|K(z,\zeta) - \hat{K}(z,\zeta)\big|^2 \mathrm{d}s_\zeta\,\mathrm{d}s_z, \tag{7}$$

and computation of the integral leads to an estimate for δ by (5). Then, we can check (6) which is either satisfied and guarantees uniqueness, or not, and calculations can be repeated with a bigger N.

Let us specify behaviour of δ in N. Consider the two-dimensional case. Let $S \in C^2$, then by using Ostrowski's inequality (see, e.g. [5, p. 468]), we find

$$\frac{1}{2}\big|K(z,\zeta) - \hat{K}(z,\zeta)\big|^2 \leqslant \big|K(z,\zeta) - K(z,\zeta_i)\big|^2 + \Big|K(z,\zeta_i) - |\gamma_j|^{-1}\int_{\gamma_j} K(z,\zeta_i)\,\mathrm{d}s_z\Big|^2$$
$$\leqslant \left[\frac{|\gamma_j|}{4} + \frac{(s_z - |\gamma_j|/2)^2}{|\gamma_j|}\right]^2 \max_{\gamma_j\times\gamma_i}|K_1'|^2 + \big(s_\zeta - s_{\zeta_i}\big)^2 \max_{\gamma_j\times\gamma_i}|K_2'|^2, \text{ when } z \in \gamma_j,\ \zeta \in \gamma_i,$$

where $K_k'(z_1,z_2) = \partial_{s(z_k)}K(z_1,z_2)$ and natural parametrization is used, so that $s_z \in (0,|\gamma_i|)$ for $z = z(s_z) \in \gamma_i$, $s_\zeta \in (0,|\gamma_j|)$ for $\zeta = \zeta(s_\zeta) \in \gamma_j$, and $\zeta_i = \zeta(s_{\zeta_i})$. We choose $s_{\zeta_i} = |\gamma_i|/2$, and denote $h = \max_{i=1,2,\ldots,N}\{|\gamma_i|\}$, then by (7) and the latter inequality we have

$$\big\|T - \hat{T}; L_2(S)\big\|^2 \leqslant \frac{h^4}{6}\sum_{i,j=1}^N\left[\frac{7}{5}\max_{\gamma_j\times\gamma_i}|K_1'|^2 + \max_{\gamma_j\times\gamma_i}|K_2'|^2\right] \leqslant \frac{h^2|S|^2}{6}\left[\frac{7}{5}\max_{S\times S}|K_1'|^2 + \max_{S\times S}|K_2'|^2\right]. \tag{8}$$

In the three-dimensional case a similar estimate could be obtained by using the modification of Ostrowski's inequality derived in [1].

From the inequalities (5) and (8) it follows that $\delta = O(h)$ as $h \to 0$. If $\nu \neq \nu_k$, where ν_k are the values corresponding to trapped modes, the estimate guarantees that the inequality (6) is fulfilled for sufficiently small h (large N). The value of N, which is needed for proving uniqueness, can be quite modest (see fig. 3), but when ν approaches ν_k, δ should decay to zero, then N tends to infinity and the big values on N demand high precision of calculation. Thus, in principle the procedure allows us to decrease size of interval, which contains ν_k, to zero, but practically the process is limited by possibilities of computer system and by time.

Numerical results obtained with the suggested criterion for the two-dimensional problem are shown in fig. 2, 3. In fig. 2 we present numerical results for two ellipses with horizontal and vertical axes a and b, respectively; centres are located at a distance l from the line $x = 0$ and at a depth d.

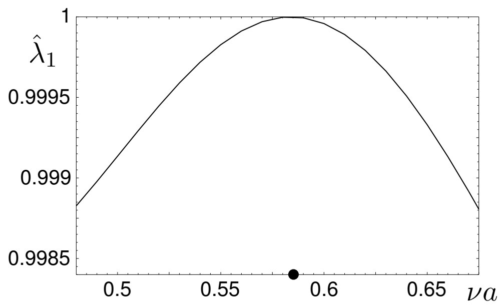

Fig. 2

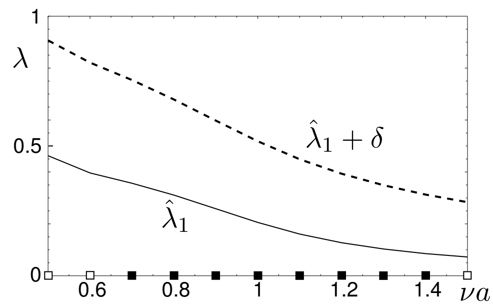

Fig. 3

Shown is the curve $\hat{\lambda}_1$ for $N = 160$, $b/a = 0.08$, $d/a = 0.25$, $l/a = 2.35$. For this value of N computed estimates of δ are much bigger than $1 - \hat{\lambda}_1$, and, hence, not shown. For the considered values of b/a, d/a, l/a existence of a trapped mode was established numerically in [7]. The corresponding value of parameter νa is marked in fig. 2 by the symbol $\bullet$.

In fig. 3 comparison with the numerical results on uniqueness, which were obtained in [6], is given. The value of δ in fig. 3 is computed by using (5) and (7). The calculations are done for two circles of a radius a, whose centres are located at a distance l from the line $x = 0$ and at a depth d. Uniqueness for $d/a = 2$, $l/a = 3$, $\nu a \in (0.5, 1.5)$ is established by the new criterion with $N = 160$. At the same time, criterion of [6] proves uniqueness for values νa marked by $\square$ and fails to prove uniqueness at points marked by $\blacksquare$.

4. Conclusion

A new approach to uniqueness in linear problems of wave–body interaction is suggested. The criterion is formulated as an inequality for the maximum eigenvalue of a self-adjoint operator combined from integral operators. A mathematically justifiable procedure for numerical verification of uniqueness property is described, numerical results and comparison with [6, 7] are presented. Future work can be directed to finding more effective estimates of the value δ defined by (5), to considering modifications of integral equations (4) and to generalizing the results to non-smooth contours.

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'A new approach to uniqueness for linear problems of wave-body interaction'

Discusser - N. Kuznetsov:

Have you tried to extend your approach to water of constant, finite depth?

Do you expect the method is applicable to the case when there are protrusions on the horizontal bottom?

Reply:

I have not tried, but as far as I can judge the extensions you suggest will not lead to any complications in comparison with the considered case.

Discusser - C.M. Linton:

Is it sufficient to find a single value of the discretization parameter at which $\hat{\lambda}_1 + \delta < 1$? Or do you need to establish that $\lim\limits_{N\to\infty} \hat{\lambda}_1 + \delta < 1$?

Reply:

In order to prove uniqueness it is sufficient to find out that $\hat{\lambda}_1(N) + \delta(N) < 1$ for a single value of discretization parameter N. And it is guaranteed that for $\nu \neq \nu_k$ (ν_k are values corresponding to trapped modes) the inequality $\hat{\lambda}_1(N) + \delta(N) < 1$ holds for sufficiently large N.