Approximations to wave scattering by steep topography

R. Porter
School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK.
richard.porter@bris.ac.uk

D. Porter
Department of Mathematics, University of Reading, Whiteknights, Reading RG6 6AX, UK.
d.porter@rdg.ac.uk

Introduction

This paper describes a method for approximating the scattering of linear surface gravity waves on water of varying depth in two dimensions. The essential difficulty in the scattering problem is the application of the Neumann condition, representing zero normal flow, on a bed profile of general shape. Fitz-Gerald (1976) showed that this difficulty is removed by conformally mapping the fluid domain into a strip of uniform width in such a way that the bed transforms into a coordinate line. The mapping therefore allows the bed condition to be satisfied exactly, but it complicates the free surface condition by introducing a variable coefficient there. However, this coefficient is relatively slowly-varying for steep and even discontinuous bed profiles, a feature exploited by Evans & Linton (1994), who used a piece-wise constant approximation of the transformed free surface condition to obtain accurate solutions in a number of test problems.

Here we take a different approach and, by analogy with the mild-slope approach familiar for slowly-varying bed profiles in the original problem, we seek an approximate solution of the transformed problem in the form of a slowly-modulating propagating wave. Optimising the use of this approximation by means of a variational principle leads to a second order ordinary differential equation for the amplitude of the modulated wave. The differential equation can be solved numerically and also analytically, in the form of a rapidly convergent infinite series that leads to simple explicit formulae for the scattered wave amplitudes. Fitz-Gerald (1976), who used an integral equation method to solve the transformed problem, demonstrated that the approach has wide applicability by presenting several families of mappings covering a range of bedforms. The approach described here is even more versatile, as small amplitude and slow variations about the topography that is transformed can be incorporated by a straightforward extension of the theory.

Formulation and approximation

We use cartesian coordinates x and y, where y is directed vertically upwards, and we denote the bed by $y = -h(x)$, where it is assumed that

$$h(x) \to \begin{cases} h_0 & (x \to -\infty), \\ h_1 & (x \to \infty). \end{cases}$$

The usual hypotheses of linear wave wave theory and the removal of a harmonic time dependence with angular frequency ω lead to equations for the spatially varying part $\phi(x, y)$ of the velocity potential in the familiar form

$$\phi_{xx} + \phi_{yy} = 0 = 0 \quad (-h(x) < y < 0), \qquad \phi_y - K\phi = 0 \quad (y = 0), \qquad \phi_n = 0 \quad (y = -h(x)), \qquad (1)$$

where ϕ_n denotes the normal derivative of ϕ on $y = -h(x)$ and $K = \omega^2/g$.

We follow Fitz-Gerald (1976) by using the conformal mapping $z = F(\zeta)$, where $z = x + \mathrm{i}y$ and $\zeta = \xi + \mathrm{i}\eta$, to transform the domain $-\infty < x < \infty, -h(x) < y < 0$ into the domain $-\infty < \xi < \infty, -1 < \eta < 0$. Writing $\phi(x, y) = \varphi(\xi, \eta)$, the boundary value problem (1) becomes

$$\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0 \quad (-1 < \eta < 0), \qquad \varphi_\eta - f(\xi)\varphi = 0 \quad (\eta = 0), \qquad \varphi_\eta = 0 \quad (\eta = -1), \qquad (2)$$

in which $f(\xi) = KF'(\xi)$ is real-valued, positive and continuous and the final equation represents the transformed bed condition. We assume that

$$f(\xi) = \begin{cases} f_0 + \mathrm{O}(\mathrm{e}^{\delta_0|\xi|}) & (\xi \to -\infty), \\ f_1 + \mathrm{O}(\mathrm{e}^{-\delta_1\xi}) & (\xi \to \infty), \end{cases}$$

where $f_i = Kh_i$ and $\delta_i > 0$ for $i = 0, 1$, which applies to a wide range of transformations.

Now with $f(\xi) = f_i$ in (2), separation of variables gives the propagating wave solutions

$$\varphi_i(\xi,\eta) = \mathrm{e}^{\pm\mathrm{i}\kappa_i\xi} v_i(\eta), \qquad v_i(\eta) = c(\kappa_i)\cosh\kappa_i(\eta+1), \qquad c(\kappa) = \Big\{ \int_{-1}^{0} \cosh^2\kappa(\eta+1)\,\mathrm{d}\eta \Big\}^{-1/2},$$

c being the normalising factor for v and κ_i the positive, real root of the dispersion relation

$$f_i = \kappa\tanh\kappa.$$

This calculation shows that the scattering problem is completed by adding the far-field conditions

$$\varphi(\xi,\eta) \sim \begin{cases} \{A_0\mathrm{e}^{\mathrm{i}\kappa_0\xi} + B_0\mathrm{e}^{-\mathrm{i}\kappa_0\xi}\}\, c(\kappa_0)\cosh\kappa_0(\eta+1) & (\xi \to -\infty), \\ \{A_1\mathrm{e}^{-\mathrm{i}\kappa_1\xi} + B_1\mathrm{e}^{\mathrm{i}\kappa_1\xi}\}\, c(\kappa_1)\cosh\kappa_1(\eta+1) & (\xi \to \infty), \end{cases} \tag{3}$$

to (2), where A_0 and A_1 are prescribed incident wave amplitudes and B_0 and B_1 are the scattered wave amplitudes that we seek. A straightforward transformation of (3) gives the corresponding far-field conditions that apply to (1) but these are not required here.

We now suppose that f is a slowly-varying function and approximate the solution of (2) by

$$\varphi(\xi,\eta) \approx \chi(\xi)v(\xi,\eta), \quad f(\xi) = \kappa(\xi)\tanh\kappa(\xi), \quad v(\xi,\eta) = c(\kappa(\xi))\cosh\kappa(\xi)(\eta+1),$$

where $\kappa(\xi)$ denotes the positive, real root of the local dispersion relation. We therefore approximate the dependence of φ on η locally by the eigenfunction v corresponding to the local value of f.

It remains to determine χ. Now it is easily shown that the functional

$$L(\psi) = \frac{1}{2}\int_a^b \Big\{ \int_{-1}^{0} (\psi_\xi^2 + \psi_\eta^2)\,\mathrm{d}\eta - f(\xi)(\psi^2)_{\eta=0} \Big\}\,\mathrm{d}\xi,$$

is stationary at the solution of (2) (we assume that variations $\delta\psi$ vanish at the arbitrary locations a and b, as we are not seeking to satisfy the far-field conditions at this stage). Thus approximations to the stationary point of L are also approximations to solutions of (2). In the spirit of the Rayleigh-Ritz method, we therefore set $\varphi(\xi,\eta) \approx \psi(\xi,\eta) = \chi(\xi)v(\xi,\eta)$ and ensure that L is stationary for arbitrary variations in χ (that vanish at a and b). After some manipulation, this procedure results in the differential equation

$$\chi'' + \{\kappa^2 - p_0(\kappa)f'^2\}\chi = 0, \tag{4}$$

where p_0 is a known function. The far-field conditions are satisfied exactly by the approximation from which it follows that we require the solution of (4) for which

$$\chi(\xi) \sim \begin{cases} A_0\mathrm{e}^{\mathrm{i}\kappa_0\xi} + B_0\mathrm{e}^{-\mathrm{i}\kappa_0\xi} & (\xi \to -\infty), \\ A_1\mathrm{e}^{-\mathrm{i}\kappa_1\xi} + B_1\mathrm{e}^{\mathrm{i}\kappa_1\xi} & (\xi \to \infty). \end{cases}$$

An analytic solution of (4) can be determined by adapting the method given in Porter (2003). Explicit expressions for the scattered wave amplitudes are thereby obtained in the form of infinite series in which successive terms are determined by a recursion relation. The series converge sufficiently rapidly that a good estimate is given by retaining only a small number of terms. Thus the simplest approximation to the reflected wave amplitude corresponding to a wave of unit amplitude incident from either the left or the right, $|R|_a$ say, given by taking only the leading term of the series and neglecting the term proportional to f'^2, is

$$|R|_a \approx \{1 + X^{-2}\}^{-1/2}, \quad X = \int_{-\infty}^{\infty} \Big(\frac{\kappa'}{2\kappa}\Big) \exp\Big\{ -2i\widehat{\kappa}s - 2i\int_0^s (\kappa(t) - \widehat{\kappa})dt \Big\} ds, \quad \widehat{\kappa} = \begin{cases} \kappa_0, & s < 0, \\ \kappa_1, & s > 0, \end{cases}$$

with $\kappa' = [(1 + \cosh 2\kappa)/(2\kappa + \sinh 2\kappa)]f'$.

More general topography

We suppose now that the bedform can be partitioned in the form $h(x) = h^{(s)}(x) - h^{(m)}(x)$, where $h^{(s)}$ is a 'steep' component that corresponds to a known conformal mapping, and the 'mild' component $h^{(m)} \geq 0$ is slowly-varying and satisfies $h^{(m)} \to 0$ as $|x| \to \infty$. The equations replacing (2) are

$$\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0 \quad (-d(\xi) < \eta < 0), \quad \varphi_\eta - f(\xi)\varphi = 0 \quad (\eta = 0), \quad \varphi_\eta + d'(\xi)\varphi_\xi = 0 \quad (\eta = -d(\xi)),$$

where $y = -h(x)$ maps into $\eta = -d(\xi) \geq -1$. This set of equations reduces to (1) in the case where $h(x)$ is differentiable if the original variables are restored, d is replaced by h and f by K.

We follow the previous approach and set

$$\varphi(\xi,\eta) \approx \chi(\xi)v(\xi,\eta), \quad v(\xi,\eta) = c(\kappa,d)\cosh\kappa(\xi)\cosh\kappa(\xi)(\eta + d(\xi)), \quad f(\xi) = \kappa(\xi)\tanh\{\kappa(\xi)d(\xi)\},$$

where c again denotes the normalising factor for v and κ the positive, real root of the local dispersion relation. Modifying the functional L by replacing the lower limit in the integral over η by $-d(\xi)$ and applying the Rayleigh-Ritz method again, we find in this case that χ satisfies

$$\chi'' + \left\{\kappa^2 - \tfrac{1}{2}||w||^{-2}d'' - q_0 f'^2 - q_1 d'^2 - q_2 f'd'\right\}\chi = 0,$$

where q_0, q_1 and q_2 are known functions of κ and d. This equation contains (4) and the modified mild-slope equation in the simplified form given by Porter (2003) as special cases. We note that this approximation requires the transformed bed shape d to be a slowly-varying function and, because of the 'stretching' effect of the mapping, the restriction on $h^{(m)}$ is relatively weak.

Results

Roseau (1976) identified a bed profile for which there is both an exact solution and a conformal mapping, and this can be used to rigorously analyse the error in the present approximations. Figure 1 shows four examples of the Roseau bedform for different values of the 'shoaling' parameter β (indicating the width of the transition between the two far-field depths), for the fixed value $\epsilon \equiv h_1/h_0 = 0.5$ of the 'steepness' parameter. We consider the amplitude of the reflected wave corresponding to an incident

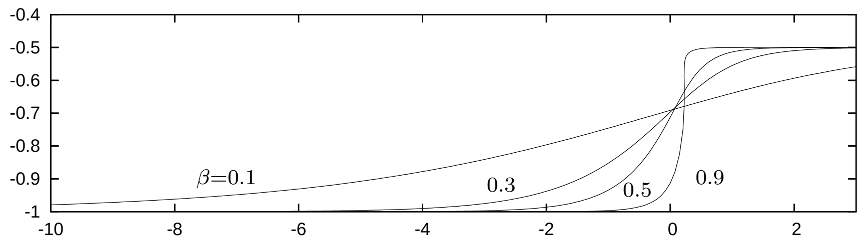

Figure 1: Examples of the Roseau profile for different values of the shoaling parameter β, with the depth ratio $\epsilon = \frac{1}{2}$.

wave (from left or right) of unit amplitude for this purpose, denoting the exact value by $|R|$. In Figure 2(i), various boundaries in the error $||R| - |R|_n|$, maximised over Kh_0, are shown in (β, ϵ) parameter space, $|R|_n$ denoting the approximate solution obtained by solving (4) numerically. The error only exceeds 0.01 when ϵ is small (corresponding to a large far-field depth ratio) and β close to unity (for which the Roseau profile overturns). Generally, the approximation is very good. Figure 2(ii) shows the error boundaries $||R| - |R|_a|$, where $|R|_a$ denotes the one-term analytic solution; the solid line corresponds to the retention of the term proportional to f'^2 and the dashed line to the formula given earlier when this term is neglected, an omission that makes little difference to the approximation. We note here that the error in $|R|_a$ is more sensitive to shoaling than steepness, but overall it is remarkably

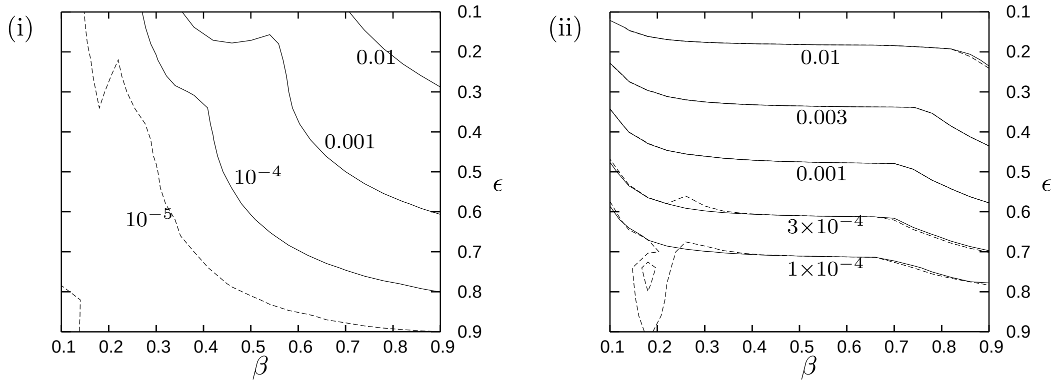

Figure 2: Error boundaries for the Roseau profile for (i) the numerical solution of (4), (ii) the one-term analytic approximation.

small given that it results from a succession of approximations. We remark that taking two terms in the infinite series form of $|R|$ gives an error distribution similar to that in Figure 2(i).

As an example of more general topography, we consider the bedform

$$d(\xi) = \begin{cases} 1 - \frac{1}{20}(1 + \cos(\pi(\xi - \sigma))), & |\xi - \sigma| \le 1, \\ 1, & |\xi - \sigma| > 1, \end{cases}$$

in the transformed plane, in conjunction with the mapping that transforms a plane slope joining horizontal beds at depths h_0 and h_1 (the familiar Booij profile). Examples of the resulting bed shapes in physical space are shown in Figure 3(i) and the corresponding numerical approximations to $|R|$ in Figure 3(ii), according to the scheme: (a) $\sigma = 0$ (long dash); (b) $\sigma = 0.5$ (short dash); (c) $\sigma = 1$; and (d) $\sigma = 1.5$. The value of $|R|$ for the unperturbed profile with $d = 1$ is shown as a solid line in Figure 3(ii). We note that the comparatively small adjustments (c) and (d) to the core profile result in the most significant variations in the approximations to $|R|$.

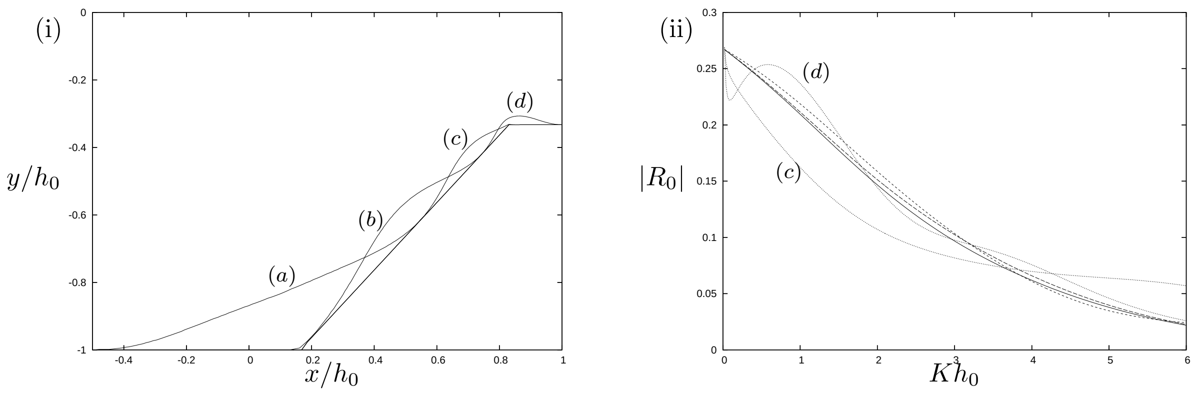

Figure 3: Perturbations about a plane sloping bed profile are shown in (i) and the corresponding approximations to $|R|$ in (ii). (The solid line in (ii) results from the unperturbed profile.)

References

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Discusser - D.V. Evans

Given a bottom topography how easy is it to find the mapping function?

Reply:

One approach is to use a piecewise linear function that is close to the topography and determine the required mapping by the Schwarz-Christoffel formula.