WATER WAVE PROBLEMS, THEIR MATHEMATICAL SOLUTION AND PHYSICAL INTERPRETATION

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1 Introduction

My first water-wave problem was in oceanography, during World War II: How do ocean waves propagate from a storm centre ? Without going into details for which see [Barber & Ursell 1948], we found that the concept of group velocity, derived from the mathematical treatment, gave us a decisive insight into the physical problem. Perhaps a detailed study of mathematical solutions might yield valuable physical insights for many problems.

Recently I have looked at my own past work in an effort to provide examples, see my Newman paper [Ursell 2006] where several problems are re-examined. I have to report that I have met with little success, The mathematics provides the solution but no physical insight.

In the present note I shall as an illustration re-work one additional problem, which I studied early in my career. It had been shown [Dean 1948] that a regular wavetrain normally incident on a submerged circular cylinder is transmitted without reflection for all values of the two independent parameters Ka and $Kf > Ka$ occurring in the problem. I gave an alternative treatment and went on to show [Ursell 1950] that the solution is unique for each pair of values; it is this uniqueness problem which will be treated here . The present treatment differs slightly from the earlier treatment.

2 The submerged circular cylinder heaving with constant frequency, uniqueness

The x-axis is horizontal, y increases with depth, $z = x + iy$. The velocity potential is $\phi(x, y)e^{i\omega t}$, where $\phi(x, y)$ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ when } y > 0 \text{ and } |z - if| > a,$$

and also satisfies the boundary conditions

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } |z - if| = a, \qquad K\phi + \frac{\partial\phi}{\partial y} = 0 \text{ on } y = 0,$$

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \quad \to \quad 0 \text{ as } y > 0 \text{ and } |z| \to \infty.$$

It will ultimately be shown that these conditions imply that $\phi(x,y) \equiv 0$. We begin by showing that any function $\phi(x,y)$ satisfying the preceding equation and boundary conditions can be continued analytically as a regular harmonic function into the image region $(y < 0, |z+if| > a)$, including the point at ∞.

From the symmetry of the equations and boundary conditions we see that $\frac{1}{2}\phi(x,y) + \frac{1}{2}\phi(-x,y)$, the even part of $\phi(x,y)$, is also a solution. We shall assume that $\phi(x,y)$ is an even function of x. (The odd part can be treated in the same way.) We consider the complex potential $w(z) = \phi(x,y) + i\psi(x,y)$. Then we know that the real part of the function

$$W(z) = Kw + i\frac{dw}{dz}$$

vanishes on $(-\infty < x < \infty,\ y = 0)$, and it follows from the Riemann Symmetry Principle that $W(z)$ can be continued by reflection, $W(x-iy) = -\overline{W(x+iy)}$, into the whole z-plane outside the two circles $|z \pm if| = a$. We can next determine $w(z)$ in the upper domain by solving the ordinary differential equation

$$Kw + i\frac{dw}{dz} = W(z)$$

along any contour starting in the lower domain. Any such contour may pass to the right or to the left of the reflected circle $|z+if| = a$; we obtain a unique continuation for w by introducing a vertical cut along $(x = 0, -\infty < y < -f)$. The discontinuity δw across the cut clearly satisfies

$$\left(K + \frac{\partial}{\partial y}\right)\delta w = 0, \ \ \text{so that} \ \ \delta w = iBe^{-Ky} \ \text{across} \ -\infty < y < -f,$$

where B is real . Let us next consider a submerged vertical wave dipole at $z = if$, its complex velocity potential $V_0(z)$ satisfies the free-surface condition, and the previous argument shows that $V_0(z)$ can be continued into the same cut plane with a discontinuity $\delta V_0 = iB_1 e^{-Ky}$, where it is readily seen that $B_1 \neq 0$. It follows that the complex potential $w_0(z) = w(z) - (B/B_1)V_0(z)$ is analytic outside the two circles $|z \pm if| = a$, and in particular when $|z| > f + a$. Thus, outside the two circles, we have the convergent expression

$$w_0(z) = \sum_{m=1}^{\infty} c_m \left(\frac{f+a}{z}\right)^m + \sum_{m=0}^{\infty} C_m \left(\frac{z}{f+a}\right)^m = f_0(z) + F_0(z).$$

In the expression

$$\begin{aligned} W(z) &= \left(K + i\frac{d}{dz}\right) w(z) \\ &= \left(K + i\frac{d}{dz}\right)\frac{B}{B_1}V_0(z) + \left(K + i\frac{d}{dz}\right) f_0(z) + \left(K + i\frac{d}{dz}\right) F_0(z) \end{aligned}$$

every term is bounded for all large z. It follows that the last term is bounded and analytic for large $|z|$ and therefore for all $|z|$, and therefore equal to 0, therefore $F_0(z) = Ce^{iKz}$ for all z, and

$$w(z) = (B/B_1)V_0(z) + Ce^{iKz} + \sum_{m=1}^{\infty} c_m \left(\frac{f+a}{z}\right)^m$$

for large $|z|$. In particular, if $w_1(z)$ now denotes the uniqueness potential, there are from energy considerations no waves at ∞ on the free surface, the first two terms on the right must be absent, and so $w_1(z)$ is seen to be analytic at ∞. We now make the conformal transformation

$$\zeta = \frac{z - iF}{z + iF} \text{ where } F = (f^2 - a^2)^{1/2},$$

then the free surface transforms into $|\zeta| = 1$, and the submerged circle transforms into $|\zeta| = s \ < 1$. We have just seen that w may be continued as far as $|\zeta| = 1/s$. Since $\psi = 0$ on $|\zeta| = s$, we see that

$$w_1 = \sum_{n=0}^{\infty} p_n \left(\zeta^n + \frac{s^{2n}}{\zeta^n} \right),$$

with real coefficients p_n, and that w_1 may be continued by inversion as far as $|\zeta| = s^2$. Thus the series for w_1 converges when $s^2 < |\zeta| < s^{-1}$, for our purpose an annulus $s(1-\epsilon) < |\zeta| \le s^{-1}(1-\epsilon)$ is sufficient. It follows that

$$p_n = O[s^n(1-\epsilon)^{-n}] \tag{2.1}$$

tends rapidly to 0 when $n \to \infty$. The boundary condition on the free surface states that

$$(1-\zeta)^2 \frac{dw}{d\zeta} + KFw \text{ is pure imaginary on } |\zeta| = 1,$$

from which it follows that

$$q_{n+1} - \left(2 - \frac{KF}{n} \right) q_n + q_{n-1} = -\frac{2KFs^{2n}}{n(1-s^{2n})} q_n, \tag{2.2}$$

where $q_n = np_n(1-s^{2n})$, thus

$$q_n = O[ns^n(1-\epsilon)^{-n}]. \tag{2.3}$$

To prove uniqueness we need to show that (2.2) and (2.3) imply that $p_n \equiv 0$ for all n.

The difference equation (2.2) has two independent solutions. We arbitrarily choose a second solution independent of $\{q_n\}$ and denote it by $\{Q_n\}$. Then there are constants A and ℓ such that

$$|Q_n| < A\ell^n \text{ for all } n.$$

For suppose that this inequality is valid for $n = N$ and $n = N - 1$, then it is valid for $n = N + 1$, if

$$|Q_{N+1}| < (2 + KF + \frac{2KFs^2}{1-s^2})A\ell^N + A\ell^{N-1} < A\ell^{N+1},$$

and therefore if

$$(2 + KF + \frac{2KFs^2}{1-s^2})\ell + 1 < \ell^2.$$

This is satisfied for all sufficiently large ℓ, and for all n if A is chosen large enough. We write

$$G(\eta) = \sum \frac{Q_n}{n} \eta^n,$$

then $G(\eta)$ has a positive radius of convergence at least equal to ℓ^{-1}. If $\ell^{-1} < 1$ we shall next show that $G(\eta)$ is convergent for all $|\eta| < 1$. For we see that

$$\begin{aligned} &(1-\eta)^2 \frac{d}{d\eta} G(\eta) + KFG(\eta) \\ = \quad & -2KF \sum \frac{s^{2n}}{n(1-s^{2n})} Q_n \eta^n + P(\eta), \text{ a polynomial in } \eta, \end{aligned}$$

where we note that the last series converges in the larger circle $|\eta| = \ell/s^2$. If $\ell/s^2 < 1$ we can now solve this differential equation for $G(\eta)$ to obtain Q_n in the larger circle, otherwise we introduce a cut from $\eta = 1$ to $\eta = \infty$. In a finite number of steps we thus continue $G(\eta)$ analytically into a domain containing $|\eta| < 1$. It follows that

$$|Q_n| < M(1+\epsilon)^n, \tag{2.4}$$

where ϵ is arbitrarily small. From the difference equation (2.2) we see at once that

$$\Pi(n) \equiv q_{n+1}Q_n - q_n Q_{n+1} = \text{ const. } \neq 0,$$

since q_n and Q_n are independent solutions of the equation. We also see from (2.1) and (2.4) that

$$\Pi(n) \to 0 \text{ when } n \to \infty.$$

This contradiction shows that we must have $q_n \equiv 0$ for all n. Uniqueness has thus been established for the even part of $\phi(x, y)$, the proof for the odd part is similar .

3 Conclusion

The first part of this proof depended essentially on analytic continuation into the upper half space in which there is no fluid and no fluid motion. It is not easy to see how such a procedure can be given a physical interpretation. I have had to use many non-physical arguments in my mathematical work.

References

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