The Neumann-Kelvin and Neumann-Michell linear flow models

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Steady free-surface potential flow about a ship that advances, with constant speed $\mathcal{U}$, in a large body of calm water of effectively infinite depth is considered. An alternative linear model, called Neumann-Michel model, to the classical Neumann-Kelvin model, is defined. The Neumann-Michel linear model accounts for dominant nonlinear free-surface effects.

Generic potential-flow representations

Nondimensional coordinates, flow velocity, and velocity potential are defined in terms of a reference length L, velocity U, and potential UL. Hereafter, $\widetilde{\mathbf{x}} = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ stands for a point inside a 3D flow region, and $\mathbf{x} = (x, y, z)$ represents a point of the boundary surface Σ of the flow region. The flow-field point $\widetilde{\mathbf{x}}$ and the boundary point $\mathbf{x}$ are associated with a Green function $G(\widetilde{\mathbf{x}};\mathbf{x})$ used to formulate boundary-integral flow representations. The flow potential at a flow-field point $\widetilde{\mathbf{x}}$ or a boundary point $\mathbf{x}$ is identified as $\widetilde{\phi}$ or ϕ, respectively. Furthermore, $d\mathcal{A}$ stands for the differential element of area at a point $\mathbf{x}$ of the boundary surface Σ, $\mathbf{n}$ is a unit vector that points inside the flow region and is normal to Σ at $\mathbf{x}$, and $\nabla = (\partial_x, \partial_y, \partial_z)$.

The potential $\widetilde{\phi} = \phi(\widetilde{\mathbf{x}})$ at a field point $\widetilde{\mathbf{x}}$ within a 3D flow region bounded by a closed boundary surface Σ is defined in terms of the boundary values of the potential ϕ and its normal derivative $\mathbf{n}\cdot\nabla\phi$ by the classical Green boundary-integral representation

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,(G\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla G) \tag{1}$$

The representation (1) defines the potential $\widetilde{\phi}$ in terms of boundary distributions of sources (with strength $\mathbf{n}\cdot\nabla\phi$) and normal dipoles (strength ϕ), and involves a Green function G and the first derivatives of G. The boundary-integral representation (1) holds for a field point $\widetilde{\mathbf{x}}$ inside the flow region, strictly outside Σ. This restriction stems from the well-known property that the potential defined by the dipole distribution in (1) is not continuous at Σ. Indeed, $\widetilde{\phi}$ on the left of (1) becomes $\widetilde{\phi}/2$ at a point $\widetilde{\mathbf{x}}$ of the boundary surface Σ (if Σ is smooth at $\widetilde{\mathbf{x}}$). The boundary-surface integral on the right of (1) is null for a point $\widetilde{\phi}$ located outside the flow region bounded by Σ.

An alternative to Green's classical potential representation (1), obtained in *Noblesse and Yang (2004)* via an integration by parts of the dipole distribution in (1), is

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,[\,G\,\mathbf{n}\cdot\nabla\phi + \mathbf{G}\cdot(\mathbf{n}\times\nabla\phi)\,] \tag{2}$$

where $\mathbf{G}$ stands for a vector Green function associated with the scalar Green function G via the relation

$$\nabla\times\mathbf{G} = \nabla G \tag{3}$$

This relation implies that $\mathbf{G}$ and G are comparable, i.e. that the behaviors of G and $\mathbf{G}$ are comparable in both the nearfield and the farfield. In particular, $\mathbf{G}$ is no more singular than G in the nearfield. Thus, the potential representation (2), which involves a Green function G and a related vector Green function $\mathbf{G}$ that is comparable to (in particular, is no more singular than) G as already noted, is weakly singular in comparison to the classical representation (1), which involves ∇G. The potential $\widetilde{\phi}$ defined by the weakly-singular representation (2) is continuous at the boundary surface Σ, whereas (1) does not define a potential $\widetilde{\phi}$ that is continuous at Σ. The relation (3) does not define a unique vector Green function $\mathbf{G}$. Indeed, if $\mathbf{G}$ satisfies (3), $\mathbf{G}+\nabla H$ also satisfies (3) for an arbitrary scalar function H. Nevertheless, the potential representation (2) defines a unique potential $\widetilde{\phi}$; see *Noblesse and Yang (2004)*. The vector Green function

$$\mathbf{G} = (\,G_y^z\,, -G_x^z\,, 0\,) \tag{4}$$

is used here. In (4), a subscript or superscript attached to G means differentiation or integration.

The potential representation

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,\{\,G\,\mathbf{n}\cdot\nabla\phi + P\,\mathbf{G}\cdot(\mathbf{n}\times\nabla\phi) - \phi\,\mathbf{n}\cdot[\,(1-P)\nabla G + \mathbf{G}\times\nabla P\,]\,\} \tag{5}$$

where $P = P(\mathbf{x}\,;\widetilde{\mathbf{x}})$ stands for a function of $\mathbf{x}$ and $\widetilde{\mathbf{x}}$, is a composite of the classical Green representation (1) and the related weakly-singular representation (2), which correspond to the special cases $P = 0$ and $P = 1$, respectively, and thus can be regarded as special cases of the more general family of potential representations (5). For a weight function P chosen so that $P \to 1$ fast enough in the nearfield, the integrands of the boundary-surface integrals in the potential representations (5) and (2) are asymptotically equivalent in the nearfield, and the potential $\widetilde{\phi}$ defined by these weakly-singular representations is continuous at Σ. Similarly, the integrands of the boundary-surface integrals in the representations (5) and (1) are asymptotically equivalent in the farfield if $P \to 0$ sufficiently rapidly in the farfield.

Application to steady linear potential flow about a ship

The generic potential-flow representation (5) is now applied to steady flow about a ship that advances at constant speed $\mathcal{U}$ in calm water. The z axis is vertical and points upward, and the mean free surface is taken as the plane $z = 0$. The x axis is chosen along the path of the ship and points toward the ship bow. The reference length L and velocity U used to nondimensionalize coordinates and the velocity potential may be chosen as the ship length and $U = \sqrt{gL}$, where g is the acceleration of gravity. An alternative reference velocity is $U = \mathcal{U}$. The closed boundary surface Σ in the boundary-integral representation (5) consists of

$$\Sigma = \Sigma_B \cup \Sigma_0 \cup \Sigma_\infty \tag{6}$$

where Σ_B stands for the mean wetted hull-surface of the ship or (more generally) a control surface that encloses the ship hull, Σ_0 is the portion of the mean free-surface plane $z = 0$ located outside the "body" surface Σ_B, and Σ_∞ is a farfield surface (e.g. the lower half of a sphere) that closes the flow domain. As already noted, the unit vector $\mathbf{n} = (\,n^x, n^y, n^z\,)$ normal to the boundary surface Σ points into the flow domain. Thus, $\mathbf{n} = (0\,,0\,,-1)$ at the free surface Σ_0.

The Green function G is presumed to vanish sufficiently rapidly in the farfield to nullify the contribution of the farfield boundary surface Σ_∞, which may be taken as a half sphere of radius a, as $a \to \infty$. The flow representation (5), with the boundary surface (6), then yields

$$\widetilde{\phi} = \widetilde{\phi}_B + \widetilde{\phi}_0 \tag{7}$$

where $\widetilde{\phi}_B$ stands for the "body component" given by (5) with Σ taken as the ship-hull surface Σ_B, and the "free-surface component" Σ_0 is defined by (5) and (4) as

$$\widetilde{\phi}_0 = -\int_{\Sigma_0} dx\,dy\,[\,G\,\phi_z - (1-P)\,G_z\,\phi - G^z_x\,(P\phi)_x - G^z_y\,(P\phi)_y\,] \tag{8}$$

The free surface Σ_0 is unbounded in (8).

Let π^ϕ and π^G stand for the functions

$$\pi^\phi = \phi_z + F^2\phi_{xx} \qquad\qquad \pi^G = G + F^2 G^z_{xx} \tag{9a}$$

where $F = \mathcal{U}/\sqrt{gL}$ is the Froude number. The integrand of the free-surface integral (8) can be expressed as $G\,\pi^\phi - A_0 + F^2 a_0$ where A_0 and a_0 are defined as

$$A_0 = (1-P)\,\pi^G_z\,\phi + (\pi^G)^z_x\,(P\phi)_x + (\pi^G)^z_y\,(P\phi)_y \tag{9b}$$

$$a_0 = [\,(1-P)\,G_x\,\phi - G\,\phi_x\,]_x + [\,G^{zz}_{xy}\,(P\phi)_y\,]_x - [\,G^{zz}_{xy}\,(P\phi)_x\,]_y$$

Here, the relation $\nabla^2 G^{zz} = 0$ was used.

Stokes' theorem then shows that (8) can be expressed as

$$\widetilde{\phi}_0 = -\int_{\Sigma_0} dx\,dy\,(\,G\,\pi^\phi - A_0\,) - F^2\int_\Gamma d\mathcal{L}\,[\,t^y(1-P)\,G_x\,\phi + G^{zz}_{xy}\,\mathbf{t}\cdot\nabla(P\phi) - t^y\,G\,\phi_x\,]$$

Here, Γ stands for the intersection curve between the body surface Σ_B and the free surface Σ_0 (in the special case when Σ_B is taken as the mean wetted ship-hull surface, rather than a control surface that encloses the ship hull, Γ is the mean ship waterline), $d\mathcal{L}$ is the differential element of arc length of Γ, and $\mathbf{t} = (\,t^x, t^y, 0\,)$ is a unit vector tangent to Γ (oriented clockwise; looking down). Substitution of the foregoing expression for $\widetilde{\phi}_0$ into (7), with (5), then yields the boundary-integral representation

$$\widetilde{\phi} = \int_{\Sigma_B} d\mathcal{A}\,\{ G\,\mathbf{n}\cdot\nabla\phi + P\,\mathbf{G}\cdot(\mathbf{n}\times\nabla\phi) - \phi\,\mathbf{n}\cdot[\,(1-P)\nabla G + \mathbf{G}\times\nabla P\,]\,\}$$
$$- F^2\int_\Gamma d\mathcal{L}\,[\,t^y(1-P)\,G_x\,\phi + G^{zz}_{xy}\,\mathbf{t}\cdot\nabla(P\phi) - \nu\,t^y G\,\phi_x\,] + \int_{\Sigma_0} dx\,dy\,(A_0 - G\pi^\phi) \tag{10}$$

where π^ϕ and A_0 are given by (9), and $\nu = 1$. The factor ν is introduced here for later use.

The velocity component ϕ_x in the line integral around Γ in (10) may be expressed in terms of the components of $\nabla\phi$ along the three orthogonal unit vectors $\mathbf{n}$, $\mathbf{t}$ and $\mathbf{d} = \mathbf{n}\times\mathbf{t}$ as

$$\phi_x = n^x\,\mathbf{n}\cdot\nabla\phi + t^x\,\mathbf{t}\cdot\nabla\phi - n^z t^y\,\mathbf{d}\cdot\nabla\phi = n^x\,\mathbf{n}\cdot\nabla\phi + t^x\,\phi_t - n^z t^y\,\phi_d \tag{11}$$

This expression defines ϕ_x in terms of the velocity component $\mathbf{n}\cdot\nabla\phi$ normal to Σ_B and the components ϕ_t and ϕ_d along the unit vectors $\mathbf{t}$ and $\mathbf{d}$ tangent to Σ_B . If Σ_B intersects the free surface orthogonally, one has $n^z = 0$ at Γ and the third component on the right of (11) is null. Substitution of (11) into (10) yields

$$\widetilde{\phi} = \widetilde{\psi} + \widetilde{\chi} \qquad \text{with} \tag{12a}$$

$$\widetilde{\psi} = \int_{\Sigma_B} d\mathcal{A}\,G\,\mathbf{n}\cdot\nabla\phi + \nu F^2\int_\Gamma d\mathcal{L}\,G\,t^y n^x\,\mathbf{n}\cdot\nabla\phi \tag{12b}$$

$$\widetilde{\chi} = \int_{\Sigma_B} d\mathcal{A}\,\{ P\,\mathbf{G}\cdot(\mathbf{n}\times\nabla\phi) - \phi\,\mathbf{n}\cdot[\,(1-P)\nabla G + \mathbf{G}\times\nabla P\,]\,\} + \int_{\Sigma_0} dx\,dy\,(A_0 - G\pi^\phi)$$
$$- F^2\int_\Gamma d\mathcal{L}\,[\,t^y(1-P)\,G_x\,\phi + G^{zz}_{xy}\,(P\phi)_t - \nu\,t^x t^y G\,\phi_t + \nu\,n^z (t^y)^2 G\,\phi_d\,] \tag{12c}$$

The potential $\widetilde{\psi}$ is defined in terms of the velocity component $\mathbf{n}\cdot\nabla\phi$ normal to Σ_B , and $\widetilde{\chi}$ is defined in terms of the potential ϕ at Σ_B and the derivatives $\mathbf{n}\times\nabla\phi$, ϕ_t and ϕ_d of ϕ along directions tangent to Σ_B .

The Neumann-Kelvin and Neumann-Michell linear models

The typical case of a surface-piercing ship, with the "body" surface Σ_B in the flow representation (12) taken as the ship-hull surface (rather than a control surface that encloses the ship), is now considered. The potential representation (10) can be expressed as

$$\widetilde{\phi} = \widetilde{\psi}_B + \widetilde{\chi}' \qquad \text{with} \tag{13a}$$

$$\widetilde{\psi}_B = \int_{\Sigma_B} d\mathcal{A}\,G\,\mathbf{n}\cdot\nabla\phi = \int_{\Sigma_B} d\mathcal{A}\,G\,n^x \tag{13b}$$

$$\widetilde{\chi}' = \int_{\Sigma_B} d\mathcal{A}\,\{ P\,\mathbf{G}\cdot(\mathbf{n}\times\nabla\phi) - \phi\,\mathbf{n}\cdot[\,(1-P)\nabla G + \mathbf{G}\times\nabla P\,]\,\}$$
$$- F^2\int_\Gamma d\mathcal{L}\,[\,t^y(1-P)\,G_x\,\phi + G^{zz}_{xy}\,(P\phi)_t - \nu\,t^y G\,\phi_x\,] + \int_{\Sigma_0} dx\,dy\,(A_0 - G\pi^\phi) \tag{13c}$$

In (13b), the ship-hull boundary condition was used.

The flow representation (13) follows from the generic potential-flow representation (5), applied to steady flow about a ship with the flow region taken as the mean flow region, bounded by the mean free surface and the mean wetted ship-hull surface. The generic potential-flow representation (5) can also be applied to the true flow region, bounded by the deformed free surface and the actual wetted ship-hull surface. In this nonlinear approach, integration over the ship-hull surface Σ_B in (13b) and (13c) must be performed up to the free surface, approximately defined by $z = F^2\phi_x$, instead of the mean free-surface plane $z = 0$. Thus, expression (13b) approximately becomes

$$\widetilde{\psi}_B \approx \int_{\Sigma_B} d\mathcal{A}\,G\,n^x + \int_\Gamma d\mathcal{L}\int_0^{F^2\phi_x} \frac{dz\,G\,n^x}{\sqrt{1-(n^z)^2}} = \int_{\Sigma_B} d\mathcal{A}\,G\,n^x - F^2\int_\Gamma d\mathcal{L}\,t^y G\,\phi_x \tag{14}$$

Here, the relation $n^x = -\,t^y\sqrt{1-(n^z)^2}$ was used. The line integral around the ship waterline Γ in (14) and the term $\nu\,t^y G\,\phi_x$ (with $\nu = 1$) in (13c) cancel out. The correction (14) for nonlinear free-surface effects then yields $\nu = 0$ in (13c).

The potential $\widetilde{\chi}'$ defined by (13c) yields additional corrections for nonlinear free-surface effects. However, these corrections are $O(\|\nabla\phi\|^2)$, whereas the correction (14) to the potential $\widetilde{\psi}$ is $O(\|\nabla\phi\|)$, i.e. is in fact linear. This significant difference stems from the Neumann-Kelvin approximation, for which n^x is not presumed to be small, and the related property (considered below) that the potential $\widetilde{\psi}_B$ dominates the potential $\widetilde{\chi}'$ in (13a). Thus, a simple correction, which accounts for dominant nonlinear free-surface effects, to the potential representations (13) or (12) is obtained by setting $\nu = 0$ in these representations. The correction (14) provides a modification of the classical Neumann-Kelvin linear model, which corresponds to $\nu = 1$, of steady flow about a ship. The linear flow model associated with $\nu = 0$ in the potential representation (12) is called Neumann-Michell model here.

Slender-ship approximations

It the boundary condition $\mathbf{n}\cdot\nabla\phi = n^x$ at the ship hull Σ_B is used in (12b) and the potential $\widetilde{\chi}$ defined by (12c) – which involves the (a priori) unknown potential ϕ and its tangential derivatives – is ignored, expression (12a) yields the approximation $\widetilde{\phi} \approx \widetilde{\psi}$ with

$$\widetilde{\psi} = \int_{\Sigma_B} d\mathcal{A}\, G\, n^x + \nu F^2 \int_{\Gamma} d\mathcal{L}\, G\, t^y (n^x)^2 \tag{15}$$

This expression, with $\nu = 1$, is the slender-ship approximation given in *Noblesse (1983)*. If one sets $\nu = 0$ in (15), one obtains the potential $\widetilde{\psi}_B$ given by (13b). The slender-ship potential (15), with $\nu = 1$ or $\nu = 0$, has been found to provide useful practical approximations, notably for hull-form optimization; e.g. *Percival et al. (2001)* and *Yang et al. (2002)*. The correction (14) for nonlinear free-surface effects and expression (11) yield the approximation (15) with $\nu = -1$. The approximations associated with $\nu = 0$ or $\nu = -1$ in (15) correspond to distributions of sources, with strength n^x, over the ship hull up to the mean free surface $z = 0$ or the (linear approximation to the) free surface $z = F^2\phi_x$, i.e. over the mean wetted ship hull or the "actual" wetted ship hull, respectively.

Numerical calculations reported in *Koch and Noblesse (1979)* and elsewhere show that the slender-ship approximation $\nu = 1$ is in better agreement with experimental measurements than the approximation $\nu = 0$ at low Froude numbers, whereas the reverse may hold at high Froude numbers; the transition Froude number is approximately equal to 0.31 and 0.32 for the two hull forms considered in *Koch and Noblesse (1979)*. This finding suggests that the usual Neumann-Kelvin model and the Neumann-Michell model might be preferable at low or high speeds, respectively, although results based on the slender-ship approximation (15) do not necessarily apply to the solution of the boundary-integral representation (12).

Concluding remarks

It can also be shown that, if $\nu = 1$, numerical cancellations occur between the surface integral over Σ_B and the line integral around Γ in (15); cancellations also occur within the integrand of the line integral around Γ in (12c). These numerical cancellations do not occur for $\nu = 0$.

This theoretical result, and the previously-noted significant numerical differences among the slender-ship approximations that correspond to $\nu = 1$ and $\nu = 0$, indicate that the usual Neumann-Kelvin linear flow model and the related Neumann-Michell model are appreciably different. The relative merits of these alternative linear flow models can only be established via comparison of experimental measurements and numerical solutions of the boundary-integral representation (12) with $\nu = 1$ (Neumann-Kelvin model) and $\nu = 0$ (Neumann-Michell model).

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