Wave scattering by a circular ice floe of variable thickness

L.G. Bennetts, N.R.T. Biggs & D. Porter
Department of Mathematics, University of Reading, Whiteknights, Reading RG6 6AX, UK.
l.g.bennetts@rdg.ac.uk, n.r.t.biggs@rdg.ac.uk, d.porter@rdg.ac.uk

Introduction

In this paper we consider the problem of water wave scattering by a circular ice floe of varying thickness and a non-zero draught, over an undulating bed. The geometrical variations are assumed to be axi-symmetric. Following Porter & Porter (2004), the problem is reformulated as a variational principle and a Rayleigh-Ritz style approximation is generated through restriction of the vertical motion to a finite dimensional space. This creates a process of vertical averaging, which eliminates the vertical coordinate from the governing equations of the approximation. By selecting a suitably large dimension to represent the vertical motion, approximations may be obtained to an arbitrary degree of accuracy. The azimuthal coordinate is also removed from calculations by implementing a Fourier cosine expansion of the circular motion, which leaves only a finite set of ordinary differential equations in the radial coordinate to be solved.

This work extends Bennetts *et al* (2007), in which numerical calculations are restricted to two-dimensional geometrical constructions. The related problem of water wave scattering by a uniform circular ice floe of zero draught has been solved previously over a finite fluid depth and a flat bed by Peter *et al* (2004) and over an infinite fluid depth by Meylan & Squire (1995).

The boundary value problem

All horizontal dependence is defined in terms of the polar coordinates (r, θ) that originate from the centre of the floe and we denote the radius of the ice floe to be R. To define the vertical structure of the geometry we use the cartesian coordinate z, which is directed upwards with its origin set to coincide with the equilibrium position of the unloaded fluid surface. For $r < R$, the undisturbed lower surface of the ice is given by the function $z = -d(r)$ and the thickness of the ice by $D = D(r)$. For $r > R$ the fluid is unloaded $(d = D = 0)$ and extends to infinity in all horizontal directions. The fluid is bounded below by a fixed, impermeable bed $z = -h(r)$, which is permitted to undulate only beneath the ice cover.

Under the usual assumptions of linear wave theory and the imposition of harmonic time dependence $e^{-i\omega t}$, the reduced velocity potential $\phi = \phi(r, \theta, z)$ must satisfy

$$\nabla^2\phi + \phi_{zz} = 0 \quad (-h < z < -d), \qquad \nabla h.\nabla\phi + \phi_z = 0 \quad (z = -h), \tag{1}$$

in which $\nabla = (1/r)(r\cos\theta\partial_r - \sin\theta\partial_\theta, r\sin\theta\partial_r + \cos\theta\partial_\theta)$. The fluid motion induces the periodic transverse oscillations $\eta(r,\theta)e^{-i\omega t}$ in the ice floe from its equilibrium position. Assuming these oscillations to be sufficiently small that linear theory applies and modelling the ice as a thin, elastic plate, we obtain the conditions

$$\nabla^2(\beta\nabla^2\eta) + \{1 - \alpha - \frac{1}{r^2}(1-\nu)(r\beta_r\partial_{rr} + r\beta_{rr}\partial_r + \beta_{rr}\partial_{\theta\theta})\}\eta - \phi = 0, \quad \nabla d.\nabla\phi + \phi_z = \kappa\eta \quad (z = -d), \tag{2}$$

which respectively represent the equation of motion of the plate and the kinematic condition at the water-ice interface. The various quantities appearing in these equations are defined as $\kappa = \omega^2/g$, $\alpha(r) = \kappa\rho_i D(r)/\rho_w$ and $\beta(r) = ED^3(r)/12\rho_w g(1-\nu^2)$, where ν is Poisson's ratio for ice, $\rho_w g\beta$ is its flexural rigidity and E is Young's modulus; ρ_i and ρ_w are respectively the densities of the ice and the fluid. Outside the circular domain occupied by the ice, conditions (2) reduce to the regular free surface condition $\phi_z = \kappa\phi$ $(z = 0)$, and the displacement function $\eta(r,\theta) \equiv \phi(r,\theta,0)$, which simply represents the free surface elevation, is redundant.

Forcing is induced by a plane wave of amplitude I, and the scattered wave is subject to the Sommerfeld radiation condition. The full solution, ϕ, in the free surface region $(r > R)$ can therefore be shown to take the form

$$\sum_{n=0}^{\infty}\Big\{I_n J_0(k_n^{(0)}r) + B_{n,0}H_0(k_n^{(0)}r) + 2\sum_{m=1}^{\infty}\mathrm{i}^m(I_n J_m(k_n^{(0)}r) + B_{n,m}H_m(k_n^{(0)}r))\cos(m\theta)\Big\}\cosh\{k_n^{(0)}(z+h)\}, \tag{3}$$

where J_m and H_m $(m = 0, \ldots)$ are respectively Bessel functions and Hankel functions of the first kind. The quantities $k_n^{(0)}$ $(n = 0, \ldots)$ are the roots of the free surface dispersion relation

$$k^{(0)} \tanh k^{(0)} h = \kappa, \tag{4}$$

arranged such that $k_0^{(0)}$ is real and positive, which provides propagating waves. The roots $k_n^{(0)}$ $(n = 1, \ldots)$ lie on the imaginary axis and are ordered in increasing magnitude, representing increasingly rapidly decaying evanescent waves. The amplitudes $B_{n,m}$ $(m, n = 0, \ldots)$ are constants to be found as part of the solution process and $I_n = I\delta_{0,n}$ is the Kronecker delta weighted by the incident amplitude.

To ease the numerical calculations, we assume that there is a positive value $\epsilon < R$ for which, within the disc $r < \epsilon$, the ice is of constant thickness and the bed is flat. The solution within this region may be expessed as

$$\phi(r, \theta, z) = \sum_{n=0}^{\infty} \{A_{n,0} \mathcal{J}_{n,0}(r) + 2 \sum_{m=1}^{\infty} \mathrm{i}^m A_{n,m} \mathcal{J}_{n,m}(r) \cos(m\theta)\} \cosh\{k_n(z + h)\}, \tag{5}$$

where $\mathcal{J}_{n,m}(r) \equiv J_m(k_n r) + \varrho_{n,1} J_m(\mu_1 r) + \varrho_{n,2} J_m(\mu_2 r)$ and the quantities k_n $(n = 0, \ldots)$ are the roots of the dispersion relation

$$(1 - \alpha + \beta k^4) k \tanh k(h - d) = \kappa, \tag{6}$$

ordered as with the roots of (4). The quantities μ_j $(j = 1, 2)$ are also roots of (6) but are (typically) complex and $\varrho_{n,j}$ $(j = 1, 2)$ are known weights. These values define waves that (typically) oscillate as well as attenuate. The amplitudes $A_{n,m}$ $(m, n = 0, \ldots)$, like $B_{n,m}$, are unspecified constants. The corresponding displacement function η may be easily obtained from (5) through (2).

Variational principle and approximation

Equations (1)-(2) and the free-surface condition $\phi_z = \kappa\phi$ $(r > R, z = 0)$ are the natural conditions of a Hamiltonian formulation of the problem as the variational principle $\delta(\mathcal{L} + \mathcal{L}^{(0)} - \mathcal{I}) = 0$, in which

$$\begin{aligned}\mathcal{L}(\psi, \chi) \; &= \; \frac{1}{2} \int_0^{2\pi} \int_0^R \Big\{ \int_{-h}^{-d} \{(\nabla\psi)^2 + \psi_z^2\} \, \mathrm{d}z + \kappa\{(1 - \alpha)\chi^2 + \beta(\nabla^2\chi)^2 - 2\chi[\psi]_{z=-d}\} \\ &-2\kappa\beta(1 - \nu)\{\frac{1}{r}(\partial_r\chi)(\partial_r^2\chi) + \frac{1}{r^2}(\partial_r^2\chi)(\partial_\theta^2\chi) - \frac{1}{r^2}(\partial_{r\theta}\chi)^2 + \frac{2}{r^3}(\partial_\theta\chi)(\partial_{r\theta}\chi) - \frac{1}{r^4}(\partial_\theta\chi)^2\}\Big\} r \, \mathrm{d}r \, \mathrm{d}\theta,\end{aligned}$$

$$\mathcal{L}^{(0)}(\psi) \; = \; \frac{1}{2} \int_0^{2\pi} \int_R^{\infty} \Big\{ \int_{-h}^{0} \{(\nabla\psi)^2 + \psi_z^2\} \, \mathrm{d}z - \kappa[\psi]_{z=0}^2 \Big\} r \, \mathrm{d}r \, \mathrm{d}\theta,$$

and

$$\mathcal{I}(\psi, u) \; = \; \int_0^{2\pi} \int_{-h}^{-d} \{([\psi]_{r=R_-} - [\psi]_{r=R_+})u\} \, \frac{1}{r} \, \mathrm{d}z \, \mathrm{d}\theta.$$

The interfacial functional $\mathcal{I}$ ensures that the continuity of fluid pressure and fluid velocity arise as natural conditions throughout the fluid depth $(-h < z < -d)$ beneath the ice edge $(r = R)$. Additionally, through the variational principle we derive ice edge conditions that dictate no flow through the submerged portion of the ice edge and vanishing of the bending moment and shearing stress. At the internal interface $(r = \epsilon)$ the stationary value must satisfy a set of essential and natural conditions related to the continuity of the variables ψ and χ at that point, where the natural conditions are provided by the variational principle.

An approximation to the stationary point (ϕ, η) may be obtained by confining the dependence of the function ψ in the vertical coordinate, z, to a finite dimensional space and seeking the stationary point over this restricted space. An approximation to the displacement function, $\chi \approx \eta$, is then produced indirectly. Motivated by (3) and (5) we employ the approximation $\phi \approx \psi_N$, where

$$\psi(r,\theta,z) \equiv \sum_{n=0}^{N} \varphi_n^{(0)}(r,\theta) w_n^{(0)}(z) \quad (r > R), \quad \psi(r,\theta,z) \equiv \sum_{n=0}^{N} \varphi_n(r,\theta) w_n(r,z) \quad (r < R). \tag{7}$$

Outside the ice-covered region, the vertical modes $w_n^{(0)}(z) = \cosh\{k_n^{(0)}(z+h)\}$ are identical to those that appear in (3). Beneath the ice $w_n(r,z) = \cosh\{k_n(r)(z+h(r))\}$ generalises the vertical modes that appear in (5) to variable geometry so that, in the annulus of varying geometry ($\epsilon < r < R$), the functions $k_n(r)$ ($n = 0, \ldots$) satisfy the dispersion relation (6) generated by the particular vertical structure at each radial value. In the disc of constant ice thickness and flat bed ($r < \epsilon$) $w_n(r,z) = w_n(z)$, $k_n(r) = k_n$ and $h(r) = h$ and the vertical modes coincide with those appearing in (5). It is therefore expected that φ_0 will represent modulated propagating waves and φ_n ($n = 1, \ldots$) the evanescent waves that are activated at the sources of scattering.

By restricting the vertical motion to that defined in (7) the variational principle generates a $2N+6$ dimensional set of governing equations, independent of z, to be satisfied by the unknown functions φ_n and $\varphi_n^{(0)}$ ($n = 0, \ldots, N$). The dimension of the approximation, N, may be increased to achieve an arbitrary degree of accuracy. This is, in effect, an application of the Rayleigh-Ritz method.

Truncated versions of the Fourier cosine series expansions of the azimuthal dependence appearing in (3) and (5) are utilized by the approximation. The dimension of these series, M, is chosen to achieve a desired degree of accuracy. It can then be shown that

$$\varphi_n^{(0)}(r,\theta) \approx I_n J_0(k_0^{(0)} r) + B_{n,0} H_0(k_n^{(0)} r) + 2\sum_{m=1}^{M} \mathrm{i}^m \{I_n J_m(k_0^{(0)} r) + B_{n,m} H_m(k_n^{(0)} r)\} \cos(m\theta), \tag{8}$$

which is a direct truncation of the vertical motion of the full linear solution. It can be further shown that, within the disc $r < \epsilon$,

$$\varphi_n(r,\theta) \approx A_{n,0}\widetilde{\mathcal{J}}_{n,0}(r) + 2\sum_{m=1}^{M} \mathrm{i}^m A_{n,m}\widetilde{\mathcal{J}}_{n,m}(r)\cos(m\theta), \;\; \widetilde{\mathcal{J}}_{n,m}(r) \equiv J_m(k_n r) + \sum_{j=1,2} \tilde{\varrho}_{n,j} J_m(\tilde{\mu}_j r), \tag{9}$$

where the values $\tilde{\varrho}_{n,j}$ and $\tilde{\mu}_j$ are dependent on the dimension of the vertical space and tend to $\varrho_{n,j}$ and μ_j respectively as N is increased. In the annulus of varying geometry we write

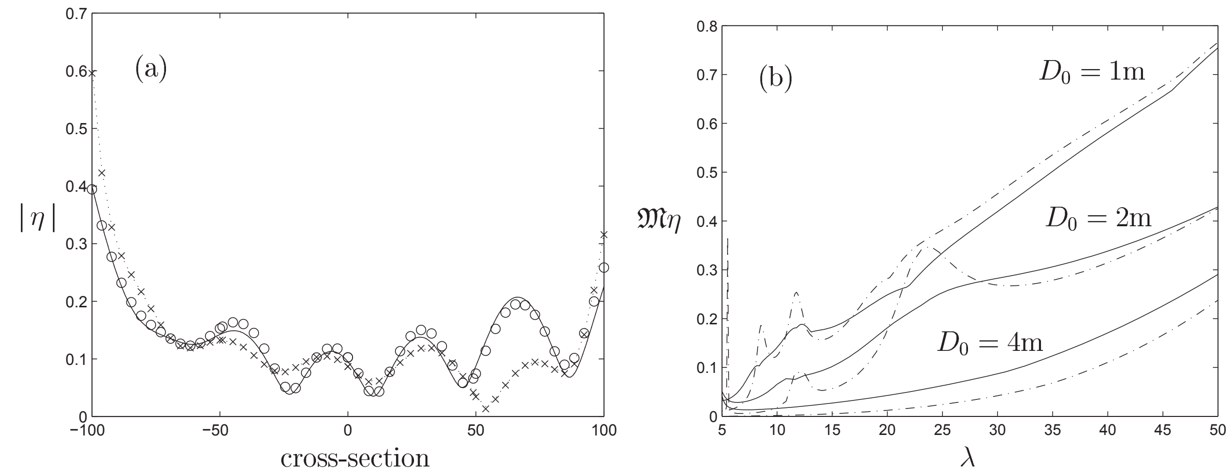
$$\varphi_n(r,\theta) \approx \varphi_{n,0}(r) + 2\sum_{m=1}^{M} \mathrm{i}^m \varphi_{n,m}(r)\cos(m\theta), \quad \chi(r,\theta) \approx \chi_0(r) + 2\sum_{m=1}^{M} \mathrm{i}^m \chi_m(r)\cos(m\theta),$$

and deduce that, for each $m = 0, \ldots, M$, the unknown functions $\varphi_{n,m}$ and χ_m must satisfy the system of ordinary differential equations

$$\sum_{i=0}^{N} \{\partial_r(r a_{n,i}\partial_r \varphi_{i,m})/r + b_{n,i}\partial_r \varphi_{i,m} + (c_{n,i} - (m/r)^2 a_{n,i})\varphi_i\} + \kappa[w_n]_{z=-d}\chi_m = 0 \quad (n = 0, \ldots, N),$$
$$\nabla_m^2(\beta\nabla_m^2\eta) + \{1 - \alpha - (1-\nu)((1/r)\partial_r(\beta_r\partial_r) - (m/r)^2\beta_{rr})\}\chi_m - \textstyle\sum_{n=0}^{N} \varphi_{n,m}[w_n]_{z=-d} = 0,$$

for $\epsilon < r < R$, where $\nabla_m^2 \equiv (1/r)\partial_r(r\partial_r) - (m/r)^2$. The coefficients $a_{n,i} = a_{n,i}(r)$, $b_{n,i} = b_{n,i}(r)$ and $c_{n,i} = c_{n,i}(r)$ ($i, n = 0, \ldots, N$) are given by integrals through the fluid depth of the modes w_n ($n = 0, \ldots, N$) and their derivatives.

We are therefore left with a set of $(M+1)$ ordinary differential equations in the radial coordinate, each of dimension $(2N+6)$. These may be numerically solved. Combining the interfacial conditions generated by the variational principle with expressions (8) and (9) provides the boundary conditions at the points $r = \epsilon, R$ for these ODEs.



Numerical results

Numerical results are shown for two example problems in the above figure. In part (a) the effect of a quadratic increase in the thickness of the ice from 10cm at its edge ($r = R$) to 1m at $r = \epsilon = R/2$ is considered. The uniform bed has $h = 20$m and the incident wavelength is chosen as $\lambda \equiv 2\pi/k_0^{(0)} = R/4$. Two problems of this form are shown. In one the thickness variation occurs on the lower surface of the ice ($\cdots$) and in the other is on the upper surface of the ice ($\times$). This is compared to a uniform floe of thickness 85cm ($-$), which is chosen to give it an identical mass to the non-uniform floes. Results for a quadratic protrusion in the bed from 20m at $r = 50$m to 10m at $r = 25$m, beneath the uniform floe, are also shown ($\circ$). We note that the two floes of varying thickness provide almost identical profiles. Despite a significant increase in the bed profile the displacements of the uniform floes are extremely similar. The relative differences between the sets of results indicates that variations to the thickness of the ice dominate over bed variations.

Part (b) displays the effect of introducing a physically correct Archimedean draught to floes of 50m radius and uniform thicknesses $D_0 = 1$m, 2m and 4m over a flat bed, $h = 20$m. The maximum value of the displacement of the floes, $\mathfrak{M}\eta \equiv \max|\eta|$, are given as functions of incident wavelength, $\lambda \in (5, 50)$m, for floes of a zero draught, $d = 0$, (solid line) and an Archimedean draught, $d = \rho_i D/\rho_w$, (dot-dash). As the floes become thicker, the displacement they experience decreases which is interpreted as a greater resistance to the incident wave. We note that the curves are monotonically increasing functions of wavelength for the longer incident waves. For shorter incident waves, this monotonicity is broken by the occurrence of local maxima. These maxima are more prevalent for the floes that incorporate a non-zero draught and thinner floes. At shorter wavelengths, there is a tendency for the floes of a non-zero draught to be displaced with a smaller magnitude than the corresponding floes of a zero draught, a tendency that is disrupted only by the occurrence of the maxima at short incident wavelengths. The point at which the floes of a non-zero draught begin to experience a smaller displacement is relative to the thickness of the floe, so that, the thicker the floe, the longer the incident wavelength that produces this behaviour. Further results will be presented at the workshop.

References

Bennetts, L. G., Biggs, N. R. T. & Porter, D. 2007 A multi-mode approximation to wave scattering by ice sheets of varying thickness *J. Fluid Mech.* In press.

Meylan, M. H. & Squire, V. A. 1995 Response of a circular ice floe to ocean waves. *J. Geophys. Res.* **101**, 8869–8884.

Peter, M. A., Meylan, M. H. & Chung, H. 2004 Wave scattering by a circular elastic plate in water of finite depth: a closed form solution. *IJOPE* **14(2)**, 81–85.

Porter, D. & Porter, R. 2004 Approximations to wave scattering by an ice sheet of variable thickness over undulating topography, *J. Fluid Mech.* **509,** 145-179.