On transformation of flexural gravity waves

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1 Introduction

In the present paper, flexural gravity wave transformation due to a heterogeneous ice-sheet floating on the free surface with change in bottom topography is analyzed in finite water depth. In case of flexural gravity waves, the wave energy density is defined as a combination of strain energy of the ice sheet, the potential energy and kinetic energy. Using the expression for wave energy density, energy relation of a general type is derived based on law of conservation of energy flux and alternately by the application of Green's second identity. Using shallow water approximation theory, explicit expressions for reflection and transmission coefficients, and the shoaling coefficient are derived by the help of continuity of the surface elevation and the law of conservation of energy flux. These analytic results will be of immense importance in the verification of the computational results obtained by various numerical methods.

2 The general boundary value problem

Transformations of flexural gravity waves take place due to various physical processes which are having adverse effects on wave characteristics and the floating structure (Porter and Porter(2004), Williams and Squire (2004)). In the context of the present study, emphasis is given on the flexural gravity wave transformation due to the change in bottom topography, thickness and rigidity of an infinitely extended floating ice-sheet which is modelled as an elastic plate based on Euler-Bernouli beam equation. The problem is analyzed in the two dimensional Cartesian co-ordinate system under the assumption of the linearized water wave theory. The entire fluid domain is divided into three regions based on the bottom surface topography $y = h(x)$, where $h(x)$ is defined as

$$h(x) = \begin{cases} h_1 & \text{for} \quad l \le x < \infty, & \text{(Region 1)} \\ h_0(x) & \text{for} \quad -l < x < l, & \text{(Region 0)} \\ h_2 & \text{for} \quad -\infty < x \le -l, & \text{(Region 2).} \end{cases} \tag{2.1}$$

Hereafter, subscripts $j = 0, 1, 2$ will denote the variables in the respective regions. The velocity potential $\Phi_j(x, y, t)$ is of the form $\Phi_j(x, y, t) = Re\{\phi_j(x, y)e^{-i\omega t}\}$ with $\phi_j(x, y)$ satisfying the two dimensional Laplace equation along with the ice-covered surface boundary condition

$$\Big(1 + D_j \frac{\partial^4}{\partial y^4}\Big)\frac{\partial \phi_j}{\partial y} + K_j\phi_j = 0 \ \text{ on } \ y = 0, \tag{2.2}$$

where $D_j = E_jI_j/\{\rho g - \rho_j d_j\omega^2\}$, $E_jI_j = E_jd_j^3/12(1-\nu_j^2)$, $K_j = \rho\omega^2/\{\rho g - \rho_j d_j\omega^2\}$, $E_j =$ Young's modulus, $\nu_j =$ Poisson's ratio, $\rho =$ density of water, $\rho_j =$ density of the ice-sheet, $g =$ acceleration due to gravity and $d_j =$ draft of the ice-sheet. The normal velocity vanishes on the rigid bottom boundary. Finally, the general form of the far field condition is prescribed as given by

$$\begin{aligned}\phi_1(x,y) &\sim \frac{\cosh k_{10}(h_1-y)}{\cosh k_{10}h_1}\{a_{11}e^{-ik_{10}x}+a_{12}e^{ik_{10}x}\} \quad \text{as} \quad x\to\infty \\ \phi_2(x,y) &\sim \frac{\cosh k_{20}(h_2-y)}{\cosh k_{20}h_2}\{a_{21}e^{-ik_{20}x}+a_{22}e^{ik_{20}x}\} \quad \text{as} \quad x\to-\infty\end{aligned} \tag{2.3}$$

with $a_{jm}, j, m = 1, 2$ are the far field wave amplitudes which depend upon the nature of the physical problem under consideration and the eigenvalues $k_{j0}, j = 1, 2$ are real and positive and satisfy the dispersion relations

$$K_j = (D_j k_{j0}^4 + 1)k_{j0}\tanh k_{j0}h_j, \quad j = 1, 2. \tag{2.4}$$

3 Energy relations for plane flexural gravity waves

Unlike the case of gravity waves, the average total wave energy per unit surface area associated with the plane flexural gravity waves is the sum of the average potential energy, kinetic energy and the surface energy. In the present context, the surface energy is generated due to the deflection of the floating ice sheet against the flexural rigidity of the floating ice sheet and is same as the strain energy. For a plane flexural gravity wave profile $\zeta(x,t) = Re\{He^{i(kx-\omega t)}/2\}$, the average potential energy $\mathcal{V}$, kinetic energy $\mathcal{T}$ and the surface energy $\mathcal{S}$ over one wave length are given by

$$\mathcal{V} = \frac{1}{L}\int_x^{x+L} \rho g(h+\zeta)\frac{(h+\zeta)}{2}dx = \frac{1}{16}\rho g H^2, \tag{3.1}$$

$$\mathcal{T} = \frac{1}{L}\int_x^{x+L}\int_{-\eta}^{h}\frac{1}{2}\rho\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]dxdy = \frac{1}{16}H^2(EIk^4+\rho g), \tag{3.2}$$

and

$$\mathcal{S} = \frac{EI}{2L}\int_x^{x+L}\left(\frac{\partial^2\zeta}{\partial x^2}\right)^2 dx = \frac{1}{16}H^2 EIk^4, \tag{3.3}$$

where the velocity potential $\Phi(x,y,t)$ is given by

$$\Phi(x,y,t) = Re\left\{\frac{iH(EIk^4+\rho g)}{2\rho\omega}\frac{\cosh k(h-y)}{\cosh kh}e^{i(kx-\omega t)}\right\} \tag{3.4}$$

with $h =$ water depth, $H =$ wave height and $L =$ wave length and the modified ice-thickness term $\delta_j = \rho_j d_j/\rho$ is neglected. It may be noted that the kinetic energy density is equal to the sum of the surface energy density and the potential energy density. Thus, the total energy density in case of flexural gravity waves is given by

$$\mathcal{E} = \mathcal{V} + \mathcal{T} + \mathcal{S} = \frac{1}{8}H^2(EIk^4+\rho g). \tag{3.5}$$

This is similar to the case of capillary gravity waves as discussed in Wehausen and Laitone (1960). Now, from law of conservation of energy flux, we have

$$\mathcal{E}c_g = \text{constant}, \tag{3.6}$$

where c_g is the group velocity of the flexural gravity waves. Thus, from (3.6) and (2.3), the energy relation is obtained as given by

$$\{|a_{11}|^2 - |a_{12}|^2\} = \gamma\{|a_{21}|^2 - |a_{22}|^2\}\frac{k_{20}\tanh k_{20}h_2(E_1I_1k_{10}^4+\rho g)}{k_{10}\tanh k_{10}h_1(E_2I_2k_{20}^4+\rho g)}, \tag{3.7}$$

where

$$\gamma = \frac{k_{10}\sinh 2k_{10}h_1}{k_{20}\sinh 2k_{20}h_2}\frac{(E_2I_2k_{20}^4+\rho g)2k_{20}h_2+(5E_2I_2k_{20}^4+\rho g)\sinh 2k_{20}h_2}{(E_1I_1k_{10}^4+\rho g)2k_{10}h_1+(5E_1I_1k_{10}^4+\rho g)\sinh 2k_{10}h_1}. \tag{3.8}$$

The energy relation (3.7) can be derived in an alternate manner by applying Green's second identity to the velocity potential ϕ and its complex conjugate $\bar{\phi}$. Thus, the surface energy term in the total energy density is justified as the alternate derivation of the energy relation does not require the definition of wave energy density. In particular, when $a_{22} = 0$ with $l = 0$, the boundary value problem reduces to the flexural gravity wave scattering due to an abrupt change in water depth (which is also called as a step) and structural inhomogeneity. In this case the reflection and transmission coefficients K_r and K_t are defined as

$$K_r = \left|\frac{a_{12}}{a_{11}}\right|, \qquad K_t = \left|\frac{a_{21}k_{20}\tanh k_{20}h_2}{a_{11}k_{10}\tanh k_{10}h_1}\right|, \tag{3.9}$$

and the energy relation (3.7) becomes

$$K_r^2 + \gamma K_t^2 = 1, \tag{3.10}$$

with γ as in (3.8). Using the expansion formulae as in Manam et al. (2006), this scattering problem is investigated and the above energy relation is used to check the correctness of the computational results, the details of which will be presented in the Workshop.

4 Wave transformation based on shallow water approximation

In this section, we shall analyze the effect of shoaling and reflection due to an abrupt change in bottom topography (assuming $l = 0$) in equation (2.1) under the assumption of shallow water approximation of the linearized theory of flexural gravity waves. Thus, neglecting the modified ice-thickness term $\delta_j = \rho_j d_j/\rho$, the linearized long wave equation in case of flexural gravity waves is given by (As in Sturova; 2001)

$$D_j'\frac{\partial^6\phi_j}{\partial x^6} + \frac{\partial^2\phi_j}{\partial x^2} = -\frac{\omega^2}{gh_j}\phi_j, \tag{4.1}$$

with $D_j' = E_jI_j/\rho g$ and the shallow water flexural gravity wave dispersion relation is given by $(D_j'k_{j0}^4+1)k_{j0}^2 = \omega^2/gh_j$. The phase and the group velocities c_j and c_{g_j} in case of shallow water are derived as $c_j = \sqrt{(D_j'k_{j0}^4+1)gh_j}$, $c_{g_j} = n_jc_j$, $n_j = (3D_j'k_{j0}^4+1)/(D_j'k_{j0}^4+1)$. The general solution form associated with the propagating wave profiles $\zeta_j(x,t)$ satisfying equation (4.1) are given by

$$\begin{aligned}\zeta_1(x,t) &= \frac{H_{11}}{2}e^{-i(k_{10}x+\omega t)} + \frac{H_{12}}{2}e^{i(k_{10}x-\omega t)} \quad \text{for} \quad x > 0,\\ \zeta_2(x,t) &= \frac{H_{21}}{2}e^{-i(k_{20}x+\omega t)} + \frac{H_{22}}{2}e^{i(k_{20}x-\omega t)} \quad \text{for} \quad x < 0,\end{aligned} \tag{4.2}$$

with $H_{jm},\ j,m = 1,2$ are the wave heights associated with the individual waves, which depend on the nature of the physical problem. To demonstrate the wave transformation due to the abrupt change in water depth, next, we will consider two particular physical problems as discussed below.

Case 1: In this case, we will consider the shoaling effect by assuming that there is no reflection and refraction of the propagating waves due to change in water depth. Thus, equation (4.2) is satisfied with $H_{j2} = 0$ for $j = 1, 2$. Hence, from the principle of conservation of energy flux as in equation (3.6), the ratio of the transmitted wave height to that of the incident wave height is obtained as

$$\frac{H_{21}}{H_{11}} = \left\{ \frac{\chi_1 \beta_1}{\chi_2 \beta_2} \right\}^{1/2}, \tag{4.3}$$

where $\chi_j = \rho g + 3E_j I_j k_{j0}^4$, $\beta_j = \sqrt{(\rho g + E_j I_j k_{j0}^4) h_j}$. This is analogous to Green's law for gravity waves as discussed in Dean and Dalrymple (1991).

Case 2: In this case, we consider the scattering of shallow water waves due to the change in water depth assuming waves are partially reflected by the step near $x = 0$. Here, equation (4.2) is satisfied for the propagating waves with $H_{22} = 0$. Thus, using the law of conservation of wave energy flux (3.6) along with the continuity of the surface elevation at $x = 0$, the reflection and transmission coefficients K_r and K_t are obtained as

$$K_r = \frac{\chi_1 \beta_1 - \chi_2 \beta_2}{\chi_1 \beta_1 + \chi_2 \beta_2}, \quad K_t = \frac{2\chi_1 \beta_1}{\chi_1 \beta_1 + \chi_2 \beta_2}, \tag{4.4}$$

where $K_r = H_{12}/H_{11}$, $K_t = H_{21}/H_{11}$. Here, it is observed that for $h_1 > h_2$ in the shallow water approximation, $K_r \to 1$ and $K_t \to 2$. This case refers to the pure standing waves in both the regions. On the other hand, the wave height ratio in case of no reflection as in (4.3) is due to propagation of progressive wave only (see Dean and Dalrymple (1991) for a comparison with the gravity wave relation). The numerical results will be discussed in detail during the Workshop.

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