Formulations of low-frequency QTF by $O(\Delta\omega)$ approximation

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The low-frequency quadratic transfer function (QTF) is defined as the second-order wave loads occurring at the frequency ($\Delta\omega$) equal to the difference ($\omega_1-\omega_2$) of two wave frequencies (ω_1, ω_2) in bichromatic waves. Its classical formulations are examined by an analysis based on the development of QTF in a power expansion with respect to $\Delta\omega$. The term associated with $\Delta\omega$ is then formulated and analyzed. It is shown that the new formulation can easily be implemented in usual panel methods solving the first- and second-order problems of wave diffraction and radiation.

1. Classical formulations of QTF

The quadratic transfer function (QTF) of low-frequency wave loads $\mathbf{F}(\omega_1, \omega_2)$ is composed of two distinct parts : one dependent only on the **q**uadratic products of first-order wave fields and another contributed by the second-order incoming and diffraction **p**otentials.

$$\mathbf{F}(\omega_1,\omega_2) = \mathbf{F_q}(\omega_1,\omega_2) + \mathbf{F_p}(\omega_1,\omega_2) \tag{1}$$

The first part $\mathbf{F_q}$ can be written in the way presented in Chen (2006a) :

$$\mathbf{F_q} = \frac{\rho}{2}\iint_H ds \left[(\nabla\phi_1\cdot\nabla\phi_2^*)\mathbf{n} - (\omega_1/\omega_2)\phi_{n2}^*\nabla\phi_1 - (\omega_2/\omega_1)\phi_{n1}\nabla\phi_2^*\right] - \frac{\rho\omega_1\omega_2}{2g}\oint_\Gamma d\ell\,(\phi_1\phi_2^*)\mathbf{n} \tag{2}$$

as integration over the hull H and along the waterline Γ in their mean position. In (2), ϕ stands for the first-order velocity potential and $\phi_n = \nabla\phi\cdot\mathbf{n}$ the normal derivative of ϕ on H. The subscripts ($_1$, $_2$) represent the quantities associated with the wave frequencies (ω_1, ω_2), respectively, while the superscript * indicates the complex conjugate.

The formulation (2) derived from Eq.27 in Chen (2006a) obtained by applying the two variants of Stokes' theorem given in Dai (1998) to the classical near-field (pressure-integration) formulation as in Pinkster (1980), is compact and used here to show the development of new formulations. It is directly applicable to force components in horizontal directions. The extension to other components is direct and omitted here.

The second part $\mathbf{F_p}$ is expressed in the way by Molin (1979) :

$$\mathbf{F_p} = -i(\omega_1-\omega_2)\rho\iint_H ds\left\{\phi_I^{(2)}\mathbf{n} - (\partial\phi_I^{(2)}/\partial n - \mathcal{N}_H)[\psi]\right\} + i(\omega_1-\omega_2)\frac{\rho}{g}\iint_F ds\,\mathcal{N}_F[\psi] \tag{3}$$

in which the first term in the hull integral corresponds to the second-order Froude-Krylov component contributed by the incoming wave potential $\phi_I^{(2)}$ defined by :

$$\phi_I^{(2)} = \frac{ia_1a_2Ag^2\cosh(k_1-k_2)(z+h)/\cosh(k_1-k_2)h}{g(k_1-k_2)\tanh(k_1-k_2)h - (\omega_1-\omega_2)^2}e^{i(k_1-k_2)(x\cos\beta+y\sin\beta)} \tag{4}$$

with A written :

$$A = \frac{\omega_1-\omega_2}{\omega_1\omega_2}k_1k_2(1+\tanh k_1h\tanh k_2h) + \frac{1}{2}\left(\frac{k_1^2/\omega_1}{\cosh^2 k_1h} - \frac{k_2^2/\omega_2}{\cosh^2 k_2h}\right) \tag{5}$$

where we have used the notations (a_1, a_2) and (k_1, k_2) standing for the wave amplitudes and wavenumbers associated with (ω_1, ω_2) via the dispersion equation $k_{1,2}\tanh k_{1,2}h = \omega_{1,2}^2/g$ with the waterdepth h, respectively, while the wave heading with respect to the positive x-axis is denoted by β.

The second term in the hull integral of (3) and the term defined by the integral over mean free surface F come from the application of Haskind relation and represent the contribution of the second-order diffraction potential, as shown in Molin (1979). The terms $(\mathcal{N}_H, \mathcal{N}_F)$ are the second members of the boundary conditions satisfied by the second-order diffraction potential on the hull H and the mean free surface F, respectively. They are written as :

$$2\mathcal{N}_H = (i\omega_2\mathbf{x}_2^*-\nabla\phi_2^*)\cdot(\mathbf{R}_1\wedge\mathbf{n}) - (i\omega_1\mathbf{x}_1+\nabla\phi_1)\cdot(\mathbf{R}_2^*\wedge\mathbf{n}) - (\mathbf{x}_1\cdot\nabla)\nabla\phi_2^*\cdot\mathbf{n} - (\mathbf{x}_2^*\cdot\nabla)\nabla\phi_1\cdot\mathbf{n} \tag{6}$$

and

$$\mathcal{N}_F = i(\omega_1 - \omega_2)(\nabla\phi_1 \cdot \nabla\phi^*_{P2} + \nabla\phi_P \cdot \nabla\phi^*_{I2}) - \frac{i\omega_1}{2g}\left[\phi_1(-\omega_2^2\partial_z + g\partial^2_{zz})\phi^*_{P2} + gk_2^2(1-\tanh^2 k_2h)\phi_{P1}\phi^*_{I2})\right] + \frac{i\omega_2}{2g}\left[\phi^*_2(-\omega_1^2\partial_z + g\partial^2_{zz})\phi_{P1} + gk_1^2(1-\tanh^2 k_1h)\phi^*_{P2}\phi_{I1})\right] \quad (7)$$

in which $\mathbf{x}$ is the displacement vector at a point on H and $\mathbf{R}$ the vector of rotations. In (7), ϕ_I represents the first-order potential of incoming waves while $\phi_P = \phi - \phi_I$ stands for that of perturbation including the diffraction and radiation components. Finally, $[\psi]$ in (3) represents a vector of first-order radiation potentials oscillating at the difference frequency $(\omega_1 - \omega_2)$. They satisfy the homogeneous condition :

$$\left[-(\omega_1 - \omega_2)^2 + g\partial_z\right][\psi] = 0 \quad (8)$$

on the mean free surface F and

$$\partial_n[\psi] = \mathbf{n} \quad (9)$$

on the hull H.

The full QTF (1) is composed of two parts $(\mathbf{F_q}, \mathbf{F_p})$ given by the formulations (2) and (3), respectively. The formulation (2) for $\mathbf{F_q}$ derived in Chen (2006a) is simpler than that in Pinkster (1980). The formulation (3) for $\mathbf{F_p}$ by Molin (1979) is often called *indirect* method since it provides a way to evaluate the contribution from the second-order diffraction potential through the Haskind relation such that the second-order diffraction potential is not explicitly computed.

2. Formulations of QTF by approximation of order $O(\Delta\omega)$

The application of low-frequency QTF concerns generally the computation of excitation loading to a moored system whose resonant frequencies are often less than 0.05 rad/s while wave frequencies ω are generally larger than 0.30 rad/s. Thus, the dynamic behavior of mooring systems is sensitive only to the low-frequency QTF at small values of $(\omega_1 - \omega_2)$. If we denote

$$\Delta\omega = \omega_1 - \omega_2 \quad \text{and} \quad \omega = (\omega_1 + \omega_2)/2 \quad \text{then} \quad (\omega_1, \omega_2) = (\omega + \Delta\omega/2, \omega - \Delta\omega/2) \quad (10)$$

we have :

$$k_1 - k_2 = 2S(k/\omega)\Delta\omega + O[(\Delta\omega)^2] \quad \text{with} \quad S = \frac{\sinh 2kh}{2kh + \sinh 2kh} \quad (11)$$

for $\Delta\omega \ll 1$. For small values of $\Delta\omega$, we can develop the first-order velocity potential as :

$$\phi_{1,2} = \phi(\omega) \pm \varphi(\omega)\Delta\omega/2 + O[(\Delta\omega)^2] \quad \text{with} \quad \varphi(\omega) = \partial\phi/\partial\omega \quad (12)$$

By introducing above (12) into (2), the first part of QTF is developed as :

$$\mathbf{F_q} = \mathbf{F^0_q} + \Delta\omega\, \mathbf{F^1_q} + O[(\Delta\omega)^2] \quad (13)$$

with

$$\mathbf{F^0_q} = \frac{\rho}{2}\iint_H ds\left[(\nabla\phi \cdot \nabla\phi^*)\mathbf{n} - \phi^*_n\nabla\phi - \phi_n\nabla\phi^*\right] - \frac{\rho\omega^2}{2g}\oint_\Gamma d\ell\,(\phi\phi^*)\mathbf{n} \quad (14)$$

a pure real function of ω which is nothing else than the formulation of drift loads (with a factor 2 of that usually used due to the convention here), and

$$\mathbf{F^1_q} = i\Im\left\{\frac{\rho}{2}\iint_H ds\left[(\nabla\varphi \cdot \nabla\phi^*)\mathbf{n} - (2/\omega)\phi_n\nabla\phi - \phi^*_n\nabla\varphi - \varphi_n\nabla\phi^*\right] - \frac{\rho\omega^2}{2g}\oint_\Gamma d\ell\,(\varphi\phi^*)\mathbf{n}\right\} \quad (15)$$

a pure imaginary function of ω. In (15) for $\mathbf{F^1_q}$, we have involved the terms $(\varphi, \nabla\varphi)$ which are defined as the derivative of $(\phi, \nabla\phi)$ with respect to ω in (12). Additional effort is needed to extend the classical panel method to obtain $(\varphi, \nabla\varphi)$ which will be discussed elsewhere.

Now we consider the second part $\mathbf{F_p}$ of the low-frequency QTF. It can be checked that the *horizontal* components of $\mathbf{F_p} = 0$ for $\Delta\omega = 0$ so that we may directly write :

$$\mathbf{F_p} = \Delta\omega\,\mathbf{F^1_p} + O[(\Delta\omega)^2] \quad \text{with} \quad \mathbf{F^1_p} = \mathbf{F^1_{p1}} + \mathbf{F^1_{p2}} + \mathbf{F^1_{p3}} \quad (16)$$

which are defined following (3) as :

$$\mathbf{F^1_{p1}} = -i\rho\iint_H ds\left(\phi^{(2)}_I\mathbf{n} - \frac{\partial\phi^{(2)}_I}{\partial n}[\psi]\right), \quad \mathbf{F^1_{p2}} = -i\rho\iint_H ds\,\mathcal{N}_H[\psi] \quad \text{and} \quad \mathbf{F^1_{p3}} = i(\rho/g)\iint_F ds\,\mathcal{N}_F[\psi] \quad (17)$$

It is understood that only the terms of order $O(1)$ in above equations (17) are to be taken.

We start with $\mathbf{F}^1_{\mathbf{p}1}$ which involves the second-order velocity potential of incoming waves $\phi_I^{(2)}$ defined by (4) on H. From (4), we can write :

$$\phi_I^{(2)} = -i2Cg/\Delta\omega + 4CgS(k/\omega)(x\cos\beta + y\sin\beta) + O(\Delta\omega) \tag{18}$$

in which C is given in Chen (2006b) as the complementary set-down in Stokes' waves :

$$C = -\frac{ka^2}{4}\left(\frac{4S+1-\tanh^2 kh}{4S^2kh - \tanh kh}\right) \tag{19}$$

The integration of the first term on the right hand side of (18) gives zero values to horizontal components of QTF so that :

$$\iint_H ds\, \phi_I^{(2)}\mathbf{n} = 4CgS(k/\omega)\forall(\cos\beta, \sin\beta) + O(\Delta\omega) \tag{20}$$

where $\forall$ stands for the buoyant volume. Furthermore, the derivation in the normal direction of $\phi_I^{(2)}$ can be written as :

$$\partial\phi_I^{(2)}/\partial n = 4CgS(k/\omega)(n_x\cos\beta + n_y\sin\beta) + O(\Delta\omega) \tag{21}$$

in which (n_x, n_y) are the horizontal components of the normal vector $\mathbf{n}$. The additional radiation potentials $[\psi]$ can be developed as :

$$[\psi] = [\psi]^0 + O[(\Delta\omega)^2] \tag{22}$$

with $[\psi]^0$ as the potentials of double-body flow being a pure real function. Furthermore, we may write :

$$-\iint_H ds\,(n_x\psi_x^0, n_x\psi_y^0, n_y\psi_x^0, n_y\psi_y^0) = (m_{xx}, m_{xy}, m_{yx}, m_{yy})\forall \tag{23}$$

in which $(\psi_x^0, \psi_y^0) = [\psi]^0$ as the horizontal components and $(m_{xx}, m_{xy}, m_{yx}, m_{yy})$ are called added-mass coefficients in double-body flow, dependent only on the hull geometry. Introducing (20), (21) and (23) into the expression of $\mathbf{F}^1_{\mathbf{p}1}$ in (17), we obtain :

$$\mathbf{F}^1_{\mathbf{p}1} = -i\rho g\forall 4CS(k/\omega)[(1+m_{xx})\cos\beta + m_{yx}\sin\beta, (1+m_{yy})\sin\beta + m_{xy}\cos\beta] \tag{24}$$

which is again a pure imaginary function of ω dependent on the hull geometry and wave heading.

The non-homogeneous term $\mathcal{N}_H$ on the right hand side of the boundary condition on H defined by (6) can be rewritten in a similar way :

$$\mathcal{N}_H = \mathcal{N}_H^0 + O(\Delta\omega) \quad \text{with} \quad \mathcal{N}_H^0 = \Re\{(i\omega\mathbf{x} - \nabla\phi)\wedge(\mathbf{R}^*\wedge\mathbf{n}) - (\mathbf{x}\cdot\nabla)\nabla\phi^*\cdot\mathbf{n}\} \tag{25}$$

so that

$$\mathbf{F}^1_{\mathbf{p}2} = -i\rho\iint_H ds\,\mathcal{N}_H^0[\psi]^0 \tag{26}$$

a pure imaginary function.

Finally, the non-homogeneous term $\mathcal{N}_F$ defined by (7) involved in the free-surface integral (3) can be written in the same way as $\mathcal{N}_H$:

$$\mathcal{N}_F = \mathcal{N}_F^0 + O(\Delta\omega) \quad \text{with} \quad \mathcal{N}_F^0 = \omega\Im\left\{\phi\partial_{zz}^2\phi_P^* + k^2\phi_P\phi_I^*\right\} \tag{27}$$

so that

$$\mathbf{F}^1_{\mathbf{p}3} = i(\rho/g)\iint_F ds\,\mathcal{N}_F^0[\psi]^0 \tag{28}$$

a pure imaginary function.

At some distance from the hull, we can write the first-order potential of perturbation ϕ_P as the sum of two components :

$$\phi_P = \phi_P^0 + \phi_P^H \tag{29}$$

with ϕ_P^0 a pure real function representing the evanescent component significant only in the vicinity of the hull and ϕ_P^H a complex function representing the plane wave component. They have properties :

$$\phi_P^{0*} = \phi_P^0 \quad \text{and} \quad \partial_{zz}^2\phi_P^H = k^2\phi_P^H \tag{30}$$

Introducing above (30) into (7), we obtain

$$\mathcal{N}_F^0 = \omega \Im(\phi_I + \phi_P^H) \left[\partial_{zz}^2 - k^2\right] \phi_P^0 \tag{31}$$

which decreases as rapid as $\phi_P^0 \approx O(R^{-2})$ with the radial distance R from the hull. In other words, the free-surface integral (28) can be evaluated in a limited zone around the hull.

3. Discussions and conclusions

By assuming $\Delta\omega \ll 1$, the quadratic transfer function (QTF) is developed as an expansion :

$$\mathbf{F} = \mathbf{F}^0 + \Delta\omega\,\mathbf{F}^1 + O[(\Delta\omega)^2] \quad \text{with} \quad \mathbf{F}^0 = \mathbf{F}_\mathbf{q}^0 \quad \text{and} \quad \mathbf{F}^1 = \mathbf{F}_\mathbf{q}^1 + \mathbf{F}_{\mathbf{p}1}^1 + \mathbf{F}_{\mathbf{p}2}^1 + \mathbf{F}_{\mathbf{p}3}^1 \tag{32}$$

in which $\mathbf{F}_\mathbf{q}^0$ given by (14) represents the wave drift loads, $\mathbf{F}_\mathbf{q}^1$ by (15) is also written as the quadratic product of the first-order potentials, $\mathbf{F}_{\mathbf{p}1}^1$ by (24) includes the Froude-Krylov component and part of second-order diffraction potential associated with the normal derivative of second-order potential of incoming waves, $\mathbf{F}_{\mathbf{p}2}^1$ by (26) is associated with the first-order motions and $\mathbf{F}_{\mathbf{p}3}^1$ by (28) is associated with the non-homogeneous term in the boundary condition on the free surface satisfied by the second-order diffraction potential.

The second-order low-frequency wave loads $\mathbf{F}(\omega_1, \omega_2)$ defined by (32) in bichromatic waves of frequencies (ω_1, ω_2) are composed of one component $\mathbf{F}^0(\omega)$ depending on $\omega = (\omega_1+\omega_2)/2$ and another $\Delta\omega\,\mathbf{F}^1(\omega)$ linearly proportional to $\Delta\omega = \omega_1 - \omega_2$. The striking fact that $\mathbf{F}^0(\omega)$ is a pure real function while $\mathbf{F}^1(\omega)$ a pure imaginary function is consistent with the finding in Molin & Chen (2002) in which the analysis is limited to cases of slender bodies like vertical cylinders. The usual approximation proposed by Newman (1974) largely used in practice is based on the use of $\mathbf{F}^0$ so that it is $O(1)$ approximation. This study confirms that not only the $O(1)$ approximation can underestimate largely the second-order wave loads as shown in Chen (1994) but also it provides wrong phase differences with respect to incoming waves since the complete QTF is a complex function while that by the $O(1)$ approximation is purely real.

Among components of $\mathbf{F}^1$ in (32), $\mathbf{F}_{\mathbf{p}1}^1$ defined by (24) depending on the second-order potential of incoming waves and hull geometry can be a dominant one for a body of large volume ($\forall$) and in water of small depth since $C \approx (3/4)ka^2/(kh)^3$ for $kh \to 0$ as shown in Chen (2006b). $\mathbf{F}_{\mathbf{p}2}^1$ given by (26) depending on first-order motions is simple and easy to evaluate, negligible for large wavenumbers. $\mathbf{F}_{\mathbf{p}3}^1$ expressed by (28) is represented by an integral over the free surface which can be performed in a limited area around the body since the integrand $\mathcal{N}_F^0[\psi]^0 \approx O(R^{-4})$ decreases rapidly with respect to the radial distance. Finally, the component $\mathbf{F}_\mathbf{q}^1$ defined by (15) can be evaluated directly when the terms φ as the derivative of first-order potential with respect to wave frequency are obtained. An indirect way consisting to evaluate the finite difference $\mathbf{F}_\mathbf{q}^1 = (\mathbf{F}_\mathbf{q} - \mathbf{F}_\mathbf{q}^0)/\Delta\omega$ after having obtained $\mathbf{F}_\mathbf{q}$ by (2) for bichromatic waves and drift component $\mathbf{F}_\mathbf{q}^0$ by (14) can be implemented.

The new formulation (32) of QTF by $O(\Delta\omega)$ approximation provides a novel method to evaluate the low-frequency second-order wave loads in a more accurate than $O(1)$ approximation and more efficient way (comparing to the computation of complete QTF). All the more interesting is that the reconstruction of time series of wave loads can be a simple sum of all wave components by using (32) instead of a double sum, if the complete QTF (1) is used, to account all pairs of wave interaction which is much more time-consuming.

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