Study of the Neumann-Kelvin problem for one hemisphere

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1 - Introduction

Very important in practice, notably for design, the seakeeping problem with forward speed in frequency domain is not yet completely satisfactory solved. Two major reasons can explain why the forward speed raises so many problems. First, the substantial difficulties associated with the numerical evaluation of the Green function and its gradients, and their subsequent integrations over ship-hull panels or waterline-segments. Second, the mathematical boundary value problem is perhaps not well posed.

To examin these two major difficulties, we shall consider the Neumann-Kelvin problem for one hemisphere. Indeed, the Neumann-Kelvin's Green function present exactly the same properties than the Green function of the seakeeping problem with forward speed. Moreover, the simple geometrical properties of the hemisphere will permit us to use a semi anlytical method to try to solve the problem.

Thus, a mixte formulation is used. The potential over the hemisphere is expanded into a series of Legendre functions by extension of the method used in ([5]). A Galerkin method is used to build the linear system to be solved. The coefficients associated with the wave component of the Green function are calculated by using the *integrated* Green function method. This enables us to control the acuracy of the wave influence coefficients. Rankine coefficients are calculated by a semi-analytical method based on a recurrence formula.

The above described semi-anlytical method is tested and validated through comparisons with classical analytical results like a floating hemisphere ([2]) and a submerged sphere with forward speed ([5]). Finally, the Neumann-Kelvin problem is discussed with different hypothesis on the continuity of the velocity potential between the hemisphere and its waterline on one hand and on the x-derivativ of the velocity potential $\partial\phi/\partial x$ on the other hand.

2 - Mathematical formulation

2-1 Boundary Value Problems and Integral Equations

The problems of the floating hemisphere from one hand, the submerged sphere and the hemisphere with forward speed on the other hand are solutions of the two following boundary value problems :

$$\begin{cases} \triangle_M \phi(M) = 0 \quad \text{, in the fluid domain} \\ \dfrac{\partial\phi}{\partial n}(P) = \begin{cases} -iwn_1(P) \text{ , for surge} \\ -iwn_3(P) \text{ , for heave} \\ -\partial\phi_0/\partial n \text{ , for diffraction} \end{cases} \quad , P\epsilon\Sigma \\ \lim\limits_{M\to\infty} \phi(M) = 0 \text{ at infinity} \\ \left\{ \dfrac{\partial\phi}{\partial z} - \dfrac{w^2}{g}\phi = 0 \right\}_{z=0}, \end{cases} \qquad \begin{cases} \triangle_M \phi(M) = 0 \quad \text{, in the fluid domain} \\ \dfrac{\partial\phi}{\partial n}(P) = -n_1(P) \text{ , } P\epsilon\Sigma \\ \lim\limits_{M\to\infty} \phi(M) = 0 \text{ at infinity} \\ \left\{ \dfrac{\partial\phi}{\partial z} + \dfrac{U^2}{g}\dfrac{\partial^2\phi}{\partial x^2} = 0 \right\}_{z=0}. \end{cases}$$

The velocity potentials $\phi(\vec{\xi})$ at a point $\vec{\xi}$ of the mean wetted surface of the two above boundary value problems are defined by the solution of an integral equation, which can be expressed in the following form for the particular case of the hemisphere with forward speed :

$$\frac{1}{2}\phi(\vec{\xi})+\iint_\Sigma \phi(\vec{x})\frac{\partial G}{\partial n}(\vec{\xi};\vec{x})dS(\vec{x})+F^2\int_W \{\phi(\vec{x})\frac{\partial G}{\partial x}(\vec{\xi};\vec{x})-\frac{\partial\phi}{\partial x}G(\vec{\xi};\vec{x})\}t_2 dl(\vec{x}) = \iint_\Sigma n_1(\vec{x})G(\vec{\xi};\vec{x})dS(\vec{x}) \ , \quad (1)$$

where Σ denotes the hemisphere, W the waterline, t_2 the y-component of the unit vector $\vec{t}$ tangent to the waterline, $F = U/(\sqrt{gr_0})$ the Froude number (r_0 the radius of the hemisphere) and $G(\vec{\xi};\vec{x})$ the Green function of the Neumann-Kelvin problem. Note that $\partial\phi/\partial x$ is also unknown along the waterline. To adress this problem, $\partial\phi/\partial x$ is decomposed into its tangential and normal components. For the hemisphere, we have :

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial\phi}{\partial\beta}\frac{\partial\beta}{\partial x}+\frac{\partial\phi}{\partial n}\frac{\partial n}{\partial x}\ \text{and}\ c_\theta=\frac{\partial\theta}{\partial x}=0\ ;\ c_\beta=\frac{\partial\beta}{\partial x}=-\frac{\sin\beta}{r_0}\ ;\ c_n=\frac{\partial r}{\partial x}=\cos\beta\quad \text{along } (W). \tag{2}$$

where (r,β,θ) are the spherical coordinates. The normal derivativ of the potential is known on the body by the Neumann condition. The problem of the tangential derivativ is adressed in the following section.

2-2 Semi-analytical method

Assuming the Neumann-Kelvin potential is continuous on the hemisphere, this potential can be expanded into a series of spherical harmonics by extension of the method used by Wu ([5]). Hence, we have :

$$\phi(\vec{\xi})=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_n^m p_n^m(\cos\theta_\xi)\cos(m\beta_\xi)\quad,\quad \theta_\xi\in[0;\pi/2],\quad \beta_\xi\in[0;2\pi] \tag{3}$$

where $p_n^m(x)$ represent the associated Legendre functions. Then after substituting this velocity potential expression (3) in the integral equation (1). a galerkin method is performed to build the linear system to be solved : we perform an integration on both sides of the integral equation (1) after multiplying them by the spherical harmonics. Thus, the coefficients A_n^m are solution of the following linear system :

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_n^m(C_{nj}^{mi_{\mathbf{N}}}+W_{nj}^{mi_{\mathbf{N}}})=S_j^{i_{\mathbf{N}}} \tag{4}$$

$$\text{with}\left\{\begin{array}{rcl} C_{nj}^{mi_{\mathbf{N}}} &=& C_{nj}^{mi_{\mathbf{N}}}(A)+C_{nj}^{mi_{\mathbf{N}}}(B)\\ C_{nj}^{mi_{\mathbf{N}}}(A) &=& \frac{1}{2}\iint_\Sigma\{p_n^m(\cos\theta_\xi)\cos(m\beta_\xi)\}\{p_j^{i_{\mathbf{N}}}(\cos\theta_\xi)\cos(i_{\mathbf{N}}\beta_\xi)\}dS(\vec{\xi})\\ \left(C_{nj}^{mi_{\mathbf{N}}}(B),W_{nj}^{mi_{\mathbf{N}}},S_j^{i_{\mathbf{N}}}\right) &=& \iint_\Sigma\left(C_n^m(\vec{\xi}),W_n^m(\vec{\xi}),S(\vec{\xi})\right)\{p_j^{i_{\mathbf{N}}}(\cos\theta_\xi)\cos(i_{\mathbf{N}}\beta_\xi)\}dS(\vec{\xi})\ , \end{array}\right.$$

$$\text{where}\left\{\begin{array}{rcl} C_n^m(\vec{\xi}) &=& \iint_\Sigma p_n^m(\cos\theta_x)\cos(m\beta_x)G_n(\vec{\xi};\vec{x})dS(\vec{x})\\ W_n^m(\vec{\xi}) &=& F^2p_n^m(0)\int_W\Big\{\cos(m\beta_x)G_x(\vec{\xi};\vec{x})+m\sin(m\beta_x)c_\beta G(\vec{\xi};\vec{x})\Big\}t_2dl(\vec{x})\\ S(\vec{\xi}) &=& \iint_\Sigma n_1G(\vec{\xi};\vec{x})dS(\vec{x})+F^2\int_W n_1c_nG(\vec{\xi};\vec{x})t_2dl(\vec{x}). \end{array}\right.$$

where $(i_{\mathbf{N}},j)$ are some integers. The solution can be obtained by truncating the infinite series at a finite number (N) of terms such that the solution has converged to the required accuracy,

The coefficients ($C_{nj}^{mi_{\mathbf{N}}}$, $W_{nj}^{mi_{\mathbf{N}}}$, $S_j^{i_{\mathbf{N}}}$) are decomposed according to the Neumann-Kelvin Green function Rankine (G^S) and free surface (G^F) components. The Rankine part is : $4\pi G^S(\vec{\xi};\vec{x})=-1/|\vec{\xi}-\vec{x}|+1/|\vec{\xi}-\vec{x}'|$ with $\vec{x}'=(x,y,-z)$ the mirror image of $\vec{x}$ with respect to the mean free surface plane (z=0). The free-surface part G^F is defined by the Fourier superposition of elementary waves ([3]) :

$$G^F(\vec{x};\vec{\xi})=\lim_{\varepsilon\to0^+}\iint_{(\alpha,\beta)}\frac{E(\vec{x})E^*(\vec{\xi})}{D_\varepsilon(\alpha,\beta)}d\alpha\,d\beta\quad\text{with}\left\{\begin{array}{rcl} E(\vec{x}) &=& \exp[kz+i(\alpha x+\beta y)]\\ E^*(\vec{\xi}) &=& \exp[k\zeta-i(\alpha\xi+\beta\eta)]\\ D_\varepsilon(\alpha,\beta) &=& 4\pi^2[-k+(i\varepsilon-F\alpha)^2] \end{array}\right. \tag{5}$$

where $k=\sqrt{\alpha^2+\beta^2}$ is the wavenumber ; (α,β) are the cartesian coordinates in the Fourier plane and (k,γ) the polar coordiantes. We have $(\alpha=k\cos\gamma)$ and $(\beta=k\sin\gamma)$.

2-4 Wave coefficients calculated by a integrated green function method

It is well known that numerical evaluation of the Green function and its gradients and their subsequent integrations over ship hull panels and waterline segments are a major stumbling block hinding the development of reliable and practical method. Indeed, it is shown formally by an asymptotic analysis that the source potential is singular and highly oscillatory for a field point approaching the track of the source at the free surface ([4], [1]).

To circumvent the above difficulties, we use an *integrated* green function method to calculate the wave coefficients. This method consists in performing the space integration before the Fourier integration. This approach is very well suited thanks to the representation of the velocity potential in terms of Legendre functions over the hemisphere. Thus, the space integration can be performed over the hemisphere. For instance, for the term of the left side of (1) associated with the integrations over the hemisphere and the waterline, we find the following expression :

$$\tilde{M}_{nj}^{mi_{\mathbf{N}}} = \tilde{C}_{nj}^{mi_{\mathbf{N}}}(B) + \tilde{W}_{nj}^{mi_{\mathbf{N}}} = \iint_{(\alpha,\beta)} \frac{S_1\Big[S_2(r_0;\alpha,\beta;m,n) + F^2 p_n^m(0) S_3(r_0;\alpha,\beta;m)\Big]}{D_\varepsilon(\alpha,\beta)} d\alpha\, d\beta \qquad (6)$$

where $\tilde{C}_{nj}^{mi_{\mathbf{N}}}(B)$ and $\tilde{W}_{nj}^{mi_{\mathbf{N}}}$ are the wave components of $C_{nj}^{mi_{\mathbf{N}}}(B)$ and $W_{nj}^{mi_{\mathbf{N}}}$. For instance, the spectrum function S_1 has the following expression :

$$S_1(r_0;\alpha,\beta;i_{\mathbf{N}},j) = \iint_\Sigma p_j^{i_{\mathbf{N}}}(\cos\theta_\xi)\cos(i_{\mathbf{N}}\beta_\xi)E^*(\vec{\xi})dS(\vec{\xi})$$

Note that these expressions depend only on the geometry of the body considered (hemisphere). This means that the study of the floating hemisphere (Hulme) allows us to validate the calculations of the spectrum functions associated with the surface integration (S_1 and S_2). The above expressions can be simplified by reducing the double integration over the surface into one simple integral. For instance, we give the expression of the spectrum function S_1 associated with the Galerkin method :

$$S_1(r_0;\alpha,\beta;i_{\mathbf{N}},j) = 2\pi i^{i_{\mathbf{N}}}(-1)^{i_{\mathbf{N}}} r_0^2 \cos(i_{\mathbf{N}}\gamma) \int_0^{\pi/2} p_j^{i_{\mathbf{N}}}(\cos\theta_\xi) e^{-kr_0\cos\theta_\xi} \sin\theta_\xi J_{i_{\mathbf{N}}}(kr_0\sin\theta_\xi)d\theta_\xi .$$

Note that the spectrum is equal to a product of functions in k and a very simple trigonometric function in γ where (k, γ) are the Fourier polar coordinates.

The expression of the wave component (6) still needs to be simplified because of the singularity of the dispersion function. To adress this problem, we use the following formula derived in ([3]) :

$$\iint_{(\alpha,\beta)} \frac{S(\alpha,\beta)}{D_\varepsilon(\alpha,\beta)} d\alpha\, d\beta = i\pi \sum_{D=0} \int_{D=0} \operatorname{sgn}(\alpha) \frac{S(\alpha,\beta)}{\|\nabla D\|} ds + \iint_{(\alpha,\beta)} \frac{S(\alpha,\beta)}{D(\alpha,\beta)} d\alpha\, d\beta \qquad (7)$$

After some calculations using the separation of the Fourier coordinates for the spectrum functions and the above expression (7), we obtain formulas which are reduced to only one integration in the Fourier plane with regular functions as integrands.

Thus, we have obtained expressions for the wave components very well suited for numerical calculations. Indeed, there is no singularity anymore and only a double integration (one Fourier, one physical) has to be performed instead of four before (two Fourier, two physical).

3 - Results

3-1 Validations

As we have mentionned before, the semi-analytical method presented here can be very easily applied to the floating hemisphere (no forward speed) which was treated in ([2]) and to the submerged sphere which was treated in ([5]). In the case of the floating hemisphere, we need only to replace the Neumann-Kelvin's Green function by the zero forward speed Green function. In the case of the submerged sphere, integrations for the spectrum functions have to be performed over the sphere instead of the hemisphere.

For the case of the floating hemisphere, the added mass and damping coefficients are evaluated and believed to be acurate to 4 decimal places. Because the surge and heave problems involve only some special orders of the spherical harmonics, we have also compared our results for the diffraction case with those obtained by HydroStar. The results are in a very good agreement.

For the case of the submerged sphere with forward speed, the wave resistance and lift coefficients are evaluated. The results are in a very good agreement with those obtained in ([5]).

3-2 Results for the Neumann-Kelvin problem for the hemisphere

If we assume the potential to be continuous over the hemisphere and if we apply the identity (2) along the waterline, the system is not inversible.

The main difference between the two previous cases of validation and the hemisphere with forward speed is the presence of the waterline integral along which was supposed the continuity of the potential

(3). That's why we have decided to let the potential to be discontinuous along the waterline by using the two separate decompositions :

$$\begin{cases} \phi(\vec{\xi}) & = & \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m p_n^m(\cos\theta_\xi)\cos(m\beta_\xi) & \text{pour} & \vec{\xi}\in(\Sigma) \\ \phi(\vec{\xi}) & = & \sum_{m=0}^{\infty} B_m \cos(m\beta_\xi) & \text{pour} & \vec{\xi}\in(W)\ . \end{cases}$$

The results are much better since the system is inversible and the hemisphere's potential converges rapidly for $z < -0.05$ and shows very simple variations like the double-body potential flow. However, for a region closer to the free surface, the variation of the velocity potential is getting more complex. The potential along the waterline doesn't seem to converge also. Moreover, the Fourier coeffcicients vary from one order of approximation to an other.

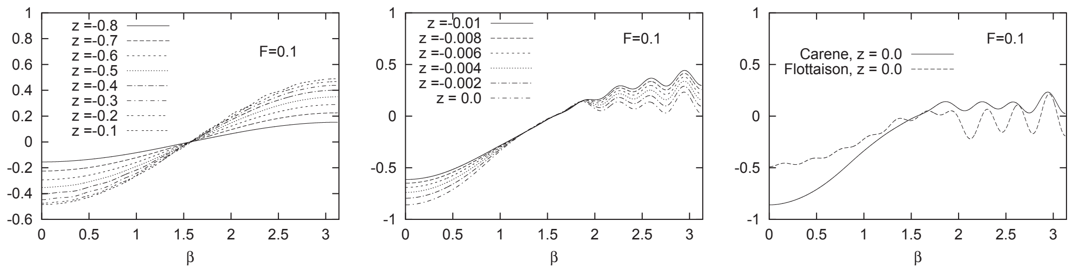

Figure 1: Potential for N=20 and $z \in [-1 : -0.1]$; Potential for N=20 and $z \in [-0.05 : 0]$; discontinuity betwwen the hemisphere's potential and waterline's potential

Because, the Neumann-Kelvin problem doesn't seem to be correctly solved, we decided to try to solve the integral equation (1) without any hypothesis like the identity we used along the waterline (2). This means that the x-derivativ $\frac{\partial\phi}{\partial x}$ of the velocity potential is considered as an unknown of our problem. The only hypothesis of this 'new' problem will be the continuity of the velocity potential between the hemispshere at $z = 0$ and the waterline. The author hopes to give definitiv results of this problem for the Workshop.

4 - Conclusion and Discussion

The Neumann Kelvin problem for a hemisphere is here considered. The velocity potential is expanded into a series of spherical harmonics over the hemisphere. A Galerkin approach is used to build the linear system to be solved. This expansion of the velocity potential into sphercial harmonics over the all hemisphere allows us to use an *integrated* Green function method to calculate very accurately the wave components of the influence coefficients. Thus, the difficult evaluation of the Neumann-Kelvin's Grren function and its subsequent integration is not anymore necessary.

This semi-analytical method is validated trough comparisons with the results obtained in ([2]) for the floating hemisphere and in([5]) for the submerged sphere with forward speed. This solid basis allows us to study more in details the usual asumptions made for the Neumann-Kelvin problem for a piercing body. These are the continuity of the potential between the hull and the waterline and the decomposition of the x-derivativ of the velocity potential (3). Indeed, the system is not inversible in the case of the usual hypothesis. Results are much better when the continuity between the hemisphere and its waterline is not imposed. But these results are not completely satisfactoty.

References

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