Theory of Scattering Frequencies Applied to Near-Trapping by Cylinders

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Introduction

At the last Workshop, Eatock Taylor *et al.* (2006) first attempted to investigate the time domain buildup to linear near-trapping by an array of cylinders, excited by a regular wave train generated by a wavemaker. The particular interest was the number of cycles required in practical terms to reach a steady state of near-trapping, following the startup of a wavemaker with an initial condition of still water. In the discussion, Meylan suggested that the buildup must be related to the imaginary part of the complex wavenumber corresponding to perfect trapping: i.e. the complex root of the coefficient matrix governing the solution of the diffraction problem for the array of cylinders. In this presentation we will provide the evidence for this suggestion, through use of the theory of scattering frequencies.

Scattering frequencies, also called resonances and sometimes complex eigenvalues, are poles of the analytic continuation of the scattering operator (or the resolvant). They occur in many linear scattering processes, not just linear water waves. In the context of water waves they have been investigated by Hazard & Lenoir (1993)and Hazard & Lenoir (2002) for the case of arbitrary two-dimensional bodies (although calculations were presented only for special cases) and Meylan (2002) where the theory was developed to the point of giving a complete description of the motion in the time-domain but only for the very simple problem of a plate floating on water of shallow draft. We apply the theory of scattering frequencies to the simple case of scattering by bottom mounted cylinders. This reduces the problem to scattering by disks with Neumann boundary conditions for the two-dimensional Helmhotz equation. It avoids many of the difficulties associated with the analytic continuation because there is only a single wavenumber which can be treated as the parameter (as opposed to the frequency) so that the analytic continuation is relatively simple. This problem has also been studied by Evans & Porter (1997) who actually calculated the scattering frequencies (without naming them) but who did not make any further calculations.

Interaction theory for cylinders

The equations for diffraction of regular waves by vertical cylinders are well known: see for example Linton & Evans (1990) whose equations are in turn based on the interaction theory of Kagemoto & Yue (1986). Using standard notation for Bessel functions etc we may write them, after truncation of the infinite series, in the form:

$$\sum_{\nu=-N}^{N} \frac{H_{\mu}^{(1)\prime}(ka)}{J_{\mu}^{\prime}(ka)} \left[\sum_{j=1,j\neq l}^{N_c} \sum_{\tau=-N}^{N} A_{\tau}^{j} H_{\tau-\nu}^{(1)}(kR_{jl}) \mathrm{e}^{\mathrm{i}(\tau-\nu)\varphi_{jl}} \right] + A_{\mu}^{l} = \sum_{\nu=-N}^{N} \frac{H_{\mu}^{(1)\prime}(ka)}{J_{\mu}^{\prime}(ka)} \tilde{D}_{\nu}^{l}. \quad (1)$$

Here N_c is the number of cylinders, each of radius a and centred at (x_c, y_c); k is the wavenumber of a wave incident at an angle χ; A_{μ}^{l} is the coefficient of the μth Fourier harmonic of scattering by the lth cylinder; φ_{jl} is the angle between the cylinders; and

$$\tilde{D}_{\nu}^{l} = e^{\mathrm{i}k(x_c \cos\chi + y_c \sin\chi)} \mathrm{e}^{\mathrm{i}\mu(\pi/2-\chi)}$$

represents the incident wave field at the lth cylinder.

This can be written as a matrix equation for the unknown vector $\mathbf{a}$ of coefficients A_{μ}^{l} :

$$\mathbf{M}(k)\,\mathbf{a} + \mathbf{a} = \mathbf{f}.$$

We do not give here an explicit expression for the coefficients of $\mathbf{M}$ for brevity, but they follow obviously from equation (1). Note that we have explicitly expressed the dependence of the matrix $\mathbf{M}$ on the parameter k. This equation is typical of scattering problems, in that we have a matrix (or operator) plus the identity. This is because the scattering process is a perturbation of the incident wave, and in the absence of scattering we have simply the identity. Exactly such an equation appears in Hazard & Lenoir (1993). The explicit solution to the scattering problem is of course given by

$$\mathbf{a} = (\mathbf{I} + \mathbf{M}(k))^{-1}\,\mathbf{f} \quad (2)$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{f}$ is the appropriate right hand side. If we allow the parameter k to become complex then the zeros of the matrix $\mathbf{I}+\mathbf{M}(k)$ (which are equivalent to the eigenvalues of the matrix $\mathbf{M}(k)$ with eigenvalue 1) are the scattering frequencies. We can study either $\mathbf{M}(k)$ (as was done in Hazard & Lenoir (1993)) or $\mathbf{I}+\mathbf{M}(k)$ (as was done in Evans & Porter (1997)) and the solution is equivalent. $(\mathbf{I}+\mathbf{M}(k))^{-1}$ is called the resolvant, so that the scattering frequencies are the singularities of the analytic continuation of the resolvant. While the matrix $\mathbf{M}(k)$ must be analytic, there is no requirement for it to be meromorphic, i.e. we have have arbitrary branch cuts, accumulation points of zeros, etc. Even in very well behaved situations (such as Meylan (2002)) there will be an infinite number of zeros. One of the challenges with the present problem is to try to understand the properties of this continuation and this remains an open question. It is possible to represent the solution in the time-domain as a sum over the scattering frequencies plus other contributions and this method is known as the singularity expansion method (SEM). We also note that, due to causality, the scattering frequencies only occur in the lower half plane, and that for very special situations a purely real zero can occur (i.e. for trapped modes). For the case of scattering by a finite array of cylinders in an unbounded domain, we do not expect to find any trapped modes.

The aim of the present work is to examine what happens when the scattering frequencies become close to the real axis. In this situation of near-trapping, there can be very large amplitudes in the response. We consider especially symmetric arrangements of identical cylinders such as were studied by Evans & Porter (1997). This allows us to make predictions of the response in the time and frequency domain. The work is still in progress and we present here only our preliminary results.

Eigenvector expansion of the scattering response

Suppose we have a scattering frequency at a complex wavenumber k_0 close to the real axis. Then we know that there is a pole of the scattering operator at k_0 so near the pole we can write the scattering operator as

$$(\mathbf{I}+\mathbf{M}(k))^{-1} \approx \frac{\mathbf{A}(k)}{k-k_0}$$

where $\mathbf{A}(k)$ is a generalisation of the residue which is connected with a projection onto the eigenspace associated with k_0. We are assuming that there is a simple zero at k_0 (in some special cases there may be a double root, requiring a relatively simple extension of the theory presented here). Finding $\mathbf{A}(k)$ is straightforward. The scattering frequency k_0 is associated with an eigenvector with zero eigenvalue. That is, we have an eigenvector $\mathbf{u}_{k_0}$ with the properties that

$$(\mathbf{I}+\mathbf{M}(k_0))\mathbf{u}_{k_0} = 0.$$

Near the point k_0 it can be shown that (Steinberg (1968))

$$(\mathbf{I}+\mathbf{M}(k_0))^{-1} \approx \frac{\mathbf{u}_{k_0}\mathbf{u}^*_{k_0^*}}{\mathbf{u}^*_{k_0^*}\mathbf{M}^{(1)}\mathbf{u}_{k_0}\,(k-k_0)}$$

where $\mathbf{u}^*_{k_0^*}$, written as a row vector, is the eigenvector of the adjoint operator with eigenvalue zero. Note that a very important property of the operator is that this occurs at k_0^*. $\mathbf{M}^{(1)}$ is given by

$$\mathbf{M}^{(1)} = \left.\frac{d}{dk}\right|_{k=k_0}\mathbf{M}(k).$$

We have therefore deduced that

$$\mathbf{A}(k) = \frac{\mathbf{u}_{k_0}\mathbf{u}^*_{k_0^*}}{\mathbf{u}^*_{k_0^*}\mathbf{M}^{(1)}\mathbf{u}_{k_0}}.$$

This means that we can approximate the solution to equation (2) near the point k_0 as

$$\mathbf{a}(k) \approx \frac{\mathbf{u}^*_{k_0^*}\mathbf{f}}{\mathbf{u}^*_{k_0^*}\mathbf{M}^{(1)}\mathbf{u}_{k_0}\,(k-k_0)}\mathbf{u}_{k_0}. \qquad (3)$$

More generally, to capture the behaviour near a set of scattering frequencies, we can write the response as a sum of such contributions from each of the corresponding poles. At present we have established only that this approximation is correct near the scattering frequencies and the properties of the more general expansion remain open. We hope to give more details at the workshop.

Results

We present here some example calculations for an array of four vertical cylinders whose centres are at the corners of a square of side $2d$, with $a/d = 0.7$. More extensive results and discussion will be provided in the presentation. The acceleration due to gravity is taken as 1. The geometry is one of the cases studied by Evans & Porter (1997), who identified a scattering frequency close to the real axis that excites a near-trapping response at $k_0 a = 3.0502+0.00195i$. These and some others are shown in Figure 1. Note that this figure shows many more scattering frequencies than were found by Evans & Porter (1997). Also, the scattering frequencies appear to be quite numerous and no pattern is discernible in their location. This is in contrast to the regular pattern in the scattering frequencies found by Meylan (2002). Finding all the zeros in a given region is a demanding computational problem. We use here a method based on a generalised bisection method described in Meylan & Gross (2003). Just as finding the eigenvalues provides an inefficient way to solve a linear system, the solution using scattering frequencies would be an inefficient way to determine the response at a given frequency (even if this were possible, which has not been established). The purpose of the scattering frequencies is to allow a deeper understanding of the system response, just as can be obtained from eigenvalues. Also note that, just as for a matrix where the difficult part is to find the eigenvalues, the computational challenge here is to find the scattering frequencies.

We compare results from the "exact" calculation based on equation (1) truncated at $N = 5$ with the approximation using the sum of contributions from near the poles we have found. In the approximation, the derivative $\mathbf{M}^{(1)}$ is obtained by simple numerical differencing.

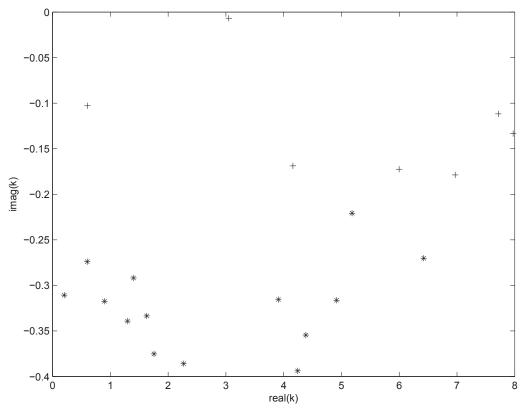

Fig. 1: Location of the zeros near the real axis.

Figure 2 shows two wave profiles evaluated at the near trapping wavenumber $k_0a = 3.0502$. The upper subplot is the modulus of the elevation (divided by incident wave amplitude) along a line bisecting two opposite sides of the square array in the case of a wave propagating parallel to that line (β=0 deg). The lower subplot is the corresponding profile along a diagonal of the square when the wave propagates along the diagonal ($\beta = 45$ deg). In the latter case a very large elevation (≈ 12) is observed on the two inner faces of the cylinders located on this diagonal, and this is fairly well predicted by the approximation. For the $\beta = 0$ deg case at this wavenumber, however, the magnification (which is largest along the line shown) is only about 2, and the approximation fails to capture this. Indeed, if only the scattering frequency corresponding to $k_0a = 3.0502$ is retained in the summation, the line labelled "'approx"' becomes a straight line at the value 1, corresponding to no scattering. This is because the corresponding eigenvector is orthogonal to the forcing function associated with a wave incident at 0 deg.

Figure 4 shows sections of the time history of elevation at the upwave point on the downwave cylinder when a wave with $k_0a = 3.0502$ at 45 deg is generated by a wavemaker, and the wave front encounters the structure (calculated using the approach described in Eatock Taylor *et al.* (2006)). It may be seen that in this case of near trapping by very closely spaced cylinders, where the imaginary part of the scattering frequency is very close to the real axis, it takes of the order of 800 periods to achieve close to steady state near trapping. This can also be seen in figure 3, where the dotted line is the envelope of the peaks of the increasing wave elevation over 830 periods. The other two lines give two approximations to this envelope. The first, labelled expfit1, is an exponential curve with argument $\mu_1 t$, where $\mu_1 = \Im\sqrt{(k_0)} = -0.00195$ for $a = 1$. The amplitude A_1 of this exponential is the near trapping result 12.23 obtained from the frequency domain analysis. The line has been fitted by a simple horizontal translation to match the temporal offset in the incoming wave. The second approximation, labelled expfit2, is an exponential fit in which three coefficients, the argument, the amplitude, and the offset, are fitted to the peaks represented by the dotted line. The relevant fitting parameters are $A_2 = 11.92, \mu_2 = 0.00190$. It is clear that the peaks are well fitted by an exponential defined by expfit2. The difference between these and the fit based on the scattering theory is most probably due to inaccuracy in the time domain numerical analysis which yields the dotted line labelled peaks. It is based on a numerical integration, which requires a very small frequency increment to capture the highly tuned response at the near trapping frequency.

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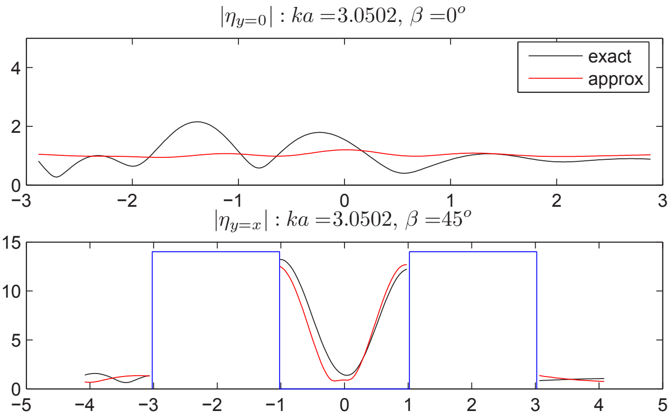

Fig. 2: Dimensionless wave profiles at near-trapping for waves at 0 and 45 degrees

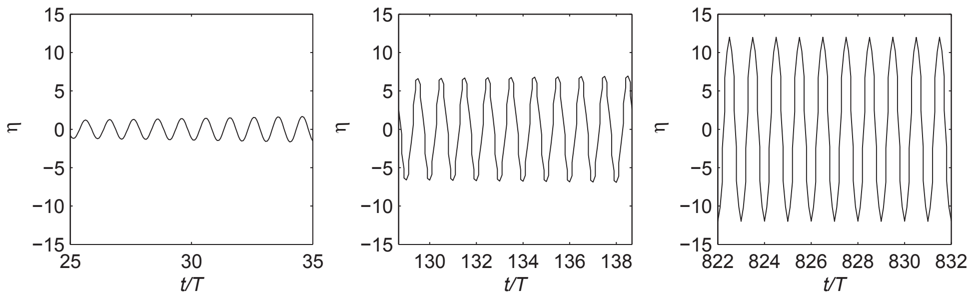

Fig. 3: Build-up of elevation at cylinder in wave generated by wavemaker at near-trapping frequency

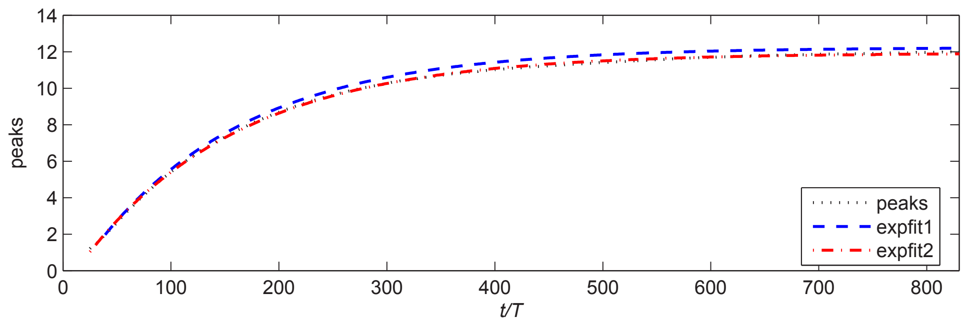

Fig. 4: Envelope of build-up to near-trapping, compared with scattering frequency theory and curve fit