Approximating near-resonant wave motion using a mechanical oscillator model

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1 Introduction

In this work, the linearised theory of water waves is used to investigate problems involving the interaction of monochromatic incident waves with fixed structures. In particular, the excitation of a single resonant mode is investigated. There are typically a number of particular incident wave frequencies for a given structural configuration at which the wave-force (and wave-motion) attains locally maximum values. These frequencies correspond to the real part of poles in the scattering and radiation potentials when extended to the complex frequency domain. In general, the positions of the poles are denoted $\omega = \omega_0 - i\epsilon$ where ω_0 is resonant frequency and ϵ is the decay rate of the resonant mode. Apart from the special case of trapped modes, the motion in the resonant mode will decay and the energy localised around the structure will radiate away from the structure until the oscillations occur at the incident wave frequency only. The complex resonant frequencies will depend on the geometry of the structure and can be estimated by examining the variation of the hydrodynamic coefficients, such as the added mass and damping, in the frequency domain.

In previous work on wave-diffraction problems, Eatock Taylor (2006) investigated the build-up of resonant wave-motion in an array of structures and considered the time for such build-up to occur. This motivated research concerning the build-up and decay of the oscillations of the fluid near a structure in a near-resonant scattering interaction. Using a time-domain solver, the response of the fluid surrounding a two-element structure was observed to follow a decaying 'beat' pattern, i.e. repeated resonant build-up was observed. The similarity of the fluid response to that of a near-resonant forced damped harmonic oscillator provided an opportunity to obtain a simple model to predict the principal properties of the motion, such as amplitude build-up, without fully solving the time-domain problem.

2 Time Domain Computations

The wave-structure interaction considered here involves the diffraction of a planar incident wave by a fixed structure with two surface piercing elements in a fluid of finite depth. The structural geometries considered thus far have been restricted to pairs of identical half-immersed semi-circular cylinders of varying sizes and spacings. The interaction was initially modelled using the two-dimensional linearised water-wave equations. To fully specify the problem, conditions for the initial state of the fluid were necessarily prescribed. The time domain solution of the initial value problem was computed using a cubic spline boundary element solver incorporating a fourth order Runge-Kutta method (see McIver, McIver & Zhang (2003) for details). This boundary element method (BEM) requires the truncation of the fluid domain via the introduction of two vertical control surfaces outside the near-field region of the structure. The solution domain is invariant in time because the linearised equations are used to describe the interaction. Despite these simplifications, these time-domain simulations are computationally intensive.

To simulate the scattering interaction described above it was necessary to numerically implement an incident wave-train in the BEM. Two methods were used to generate an incident wave-train in these time-domain computations. In the first method, initial conditions on the free-surface specified that a finite wave packet was present in the free-surface section to the left of the structures at $t = 0$. The initial wave-profile was that of a right-moving sinusoidal wave-packet of unit amplitude, with 'ramped intervals' at the front and rear of the packet present to minimise computational instabilities. The scattered waves are absorbed by the combined effect of an absorption beach (adjacent to each control surface) and the implementation of each control surface as a piston absorber.

The second method used to generate an incident wave-train was to treat the left-hand control surface as a piston wave-maker by an appropriate change of the relevant boundary condition. Therefore, a regular oscillatory term was added to the original absorption boundary condition, described in detail by McIver et al. (2003), so that the new boundary condition allowed both the generation of incident waves and the absorption of reflected waves. The absorption term in the left-hand control-surface boundary condition was retained so as to minimise computational problems due to artificial reflections. A frequency domain analysis of the wave-maker problem with this type of boundary condition was necessary to compute the appropriate control surface velocity amplitude for the generation of a unit-amplitude regular wave-train. To properly implement the wave-maker boundary condition, it was also necessary to remove the absorption beach adjacent to the left-hand control surface. This wave-maker method generates an indefinitely propagating regular wave-train after an initial transient time interval during which waves of longer wavelength and smaller amplitude occur. In both cases the regular part of the resultant incident wave-train is identical.

3 Mechanical Oscillator Model

A simple approximation of the response of the fluid between the structural elements to the incident wave can be found by modelling the interaction process as a damped harmonic oscillator undergoing an oscillatory forcing. However, this model is valid only if the incident wave excites significantly one resonant mode. If more than one mode is excited, the damped harmonic oscillator equation will not be sufficiently complex to capture even the basic features of the wave motion. Therefore, provided the incident wave of frequency ω excites a single resonant mode of frequency ω_0 such that $\omega \simeq \omega_0$ the wave-motion of the fluid in the internal free-surface section between the structures can be described using the equation

$$\ddot{\chi} + 2\epsilon\dot{\chi} + \omega_0^2\chi = F(t) \tag{1}$$

where the decay constant of the resonant mode is ϵ. The factor of 2 multiplying the $\dot{\chi}$ term is necessary to produce a decay term in the resonant component of the motion of the form $e^{-\epsilon t}$. The quantity χ can be any dynamic quantity associated with the motion of the fluid in the wave-structure interaction. Here, χ is chosen to represent the wave-elevation η at the mid-point of the free-surface between the structures and thus $\chi(0) = \dot{\chi}(0) = 0$. The predictions from this model are to be compared to the time-domain results for η at the mid-point.

The incident wave is modelled using the forcing $F(t)$ so that during the interaction of the steady-state part of the wave-train the forcing will be given by $F(t) = A\sin(\omega t + \alpha)$ where α is the phase of the forcing relative to the resonant motion. The amplitude A is scaled so that the first maximum in the beat envelope is equal to that for the time-domain computation. It is important to note that it is not possible to predict the amplitude of the response and how it relates to the incident wave since the complexity of the excitation process is beyond the scope of this model. Nevertheless, the variation of the response with time can be determined provided the function $F(t)$ approximates the incident wave forcing well. A different forcing is used to implement the waves resulting from the two generation methods described in § 2. For the initial wave-profile method the forcing function used has the form

$$F(t) = (H(t - t_1) - H(t - t_2))A\sin(\omega t + \alpha) \tag{2}$$

and for the wave-maker method the forcing is

$$F(t) = H(t - t_3)A\sin(\omega t + \alpha) \tag{3}$$

where $H(t)$ is the Heaviside function. In equation (2), $t_1 < t_2$ and both of these times depend on the initial wave-profile and the group velocity c_g whereas in equation (3), t_3 was chosen so the first envelope maximum occurred at the same time as in the time-domain results. Next, the method for calculating values of ω_0 and ϵ for a given structure is outlined.

4 Complex Resonances — Poles in the Potentials

To obtain a decaying 'beat' type response of the fluid near the structure it is necessary that the frequency of the structure's (primary) resonant mode be similar to that of the incident wave. Furthermore, the incident wave should excite significantly only one complex resonance with frequency ω_0 and no others so that the response is dominated by the forced motion and one resonant motion. This is necessary in order to apply the mechanical oscillator model to approximate the response. Therefore, the complex resonant frequency of the given structure near the incident wave frequency must be determined to ensure the previous requirements are satisfied and also to enable the computation of an approximate solution from equation (1).

The method for obtaining an estimate for the resonant frequency ω_0 and the decay constant ϵ is the same as that used by McIver et al. (2003). This requires the computation of the frequency dependence of the hydrodynamic coefficients in the neighbourhood of the resonant frequency. Thus, assuming the resonance is a vertical pumping mode, the heave-heave components of the added mass and damping coefficient matrices, denoted a_{33} and b_{33} respectively, must be computed. It should be noted that this method is only accurate in the case where the pole lies close to the real-ω axis so that a small value of ϵ gives a more accurate computation. Determining the position of the pole $\omega = \omega_0 - i\epsilon$ from the added mass and damping coefficients is possible because the existence of a pole in the heave potential ϕ_3 implies the presence of a corresponding pole in the complex force coefficient, as can be seen from the definition

$$f_{33}(\omega) = i\omega \iint_{S_B} \phi_3 n_3 dS = i\omega \left(a_{33} + \frac{ib_{33}}{\omega} \right). \tag{4}$$

Thus, if the non-dimensional forms of the added mass and damping are denoted by μ and ν respectively, the presence of a pole in the force coefficient implies that

$$\mu + i\nu \sim -\frac{\mathcal{A}}{\omega - (\omega_0 - i\epsilon)} \quad \text{as} \quad \omega \to \omega_0 - i\epsilon. \tag{5}$$

Given that $\mathcal{A} > 0$, the damping coefficient will always be positive and will have a minimum value at $\omega = \omega_0$. The added mass coefficient, on the other hand, will have a local maximum and minimum at $\omega = \omega_0 - \epsilon$ and $\omega = \omega_0 + \epsilon$ respectively. This behaviour can be observed in figure 1 where μ and ν are plotted in the neighbourhood of the resonant frequency for the particular structural geometry described next.

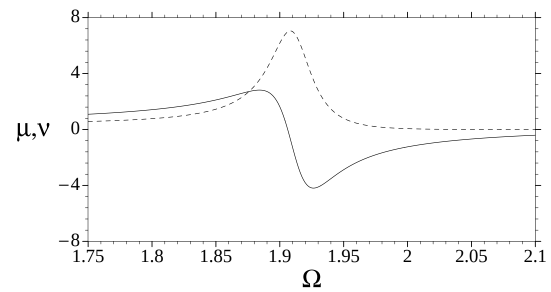

Figure 1: Variation of the added mass μ (—) and damping coefficient ν (– –) with frequency near the resonant frequency.

The results presented in figure 1 pertain to the non-dimensionalised linear water-wave equations with a structure comprising of two circular cylindrical elements with radii of 0.3 and centres positioned a distance 0.7 apart. The added mass and damping were computed at increments of $\Delta\omega = 0.01$ on the frequency interval $[1.75, 2.1]$. The maximum damping occurred at $\omega = 1.909$ and the maximum and minimum added mass occurred at the frequencies 1.885 and 1.928 respectively; thus, $\omega_0 \simeq 1.909$ and $\epsilon \simeq \frac{1.928-1.885}{2} \simeq 0.021$.

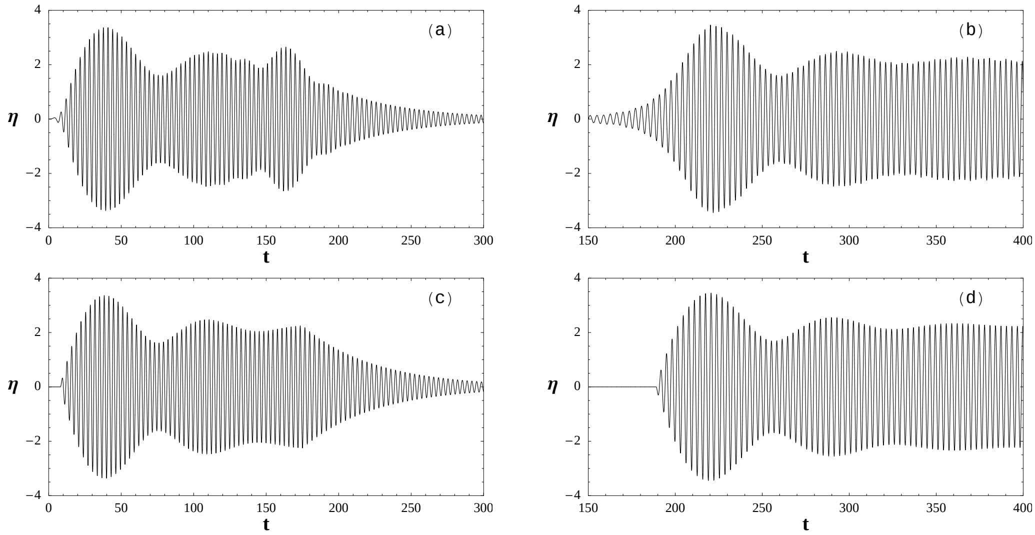

Figure 2: Free-surface elevation η as a function of time t at the mid-point between the structural elements for the incident wave imposed as an initial wave-profile ((a) and (c)) and for the incident wave generated by the wave-maker ((b) and (d)).

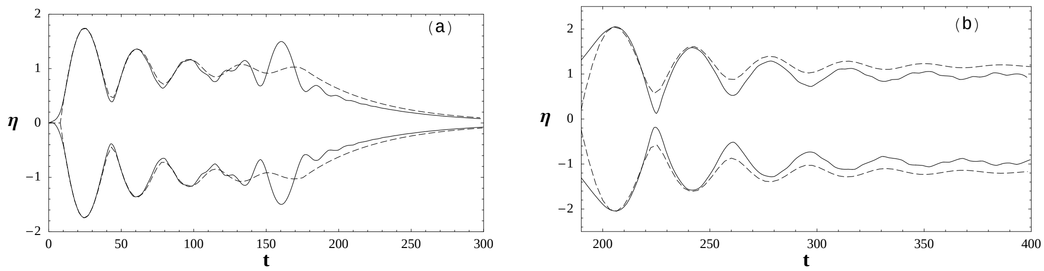

Figure 3: Free-surface elevation η as a function of time t at the mid-point between the structural elements for the initial wave profile (a) and wave-maker (b) generated incident wave trains. The mechanical oscillator model predictions (—) are compared to the BEM results (−−).

5 Results

The results of the mechanical oscillator model are compared to the time domain solution here for both types of incident wave-train. The regular part of the incident wave is specified to have a frequency of $\omega = \sqrt{4 \tanh 4}$ (corresponding to a wavenumber $k = 4$) where the time-domain equations were non-dimensionalised prior to their numerical implementation. The first set of results, shown in figure 2, correspond to the structure geometry analysed in the last section. The time-domain solutions are shown in diagrams (a) and (b) while the corresponding mechanical oscillator results are shown in diagrams (c) and (d) of figure 2. It can be seen that the mechanical oscillator results give a satisfactory approximation to the build-up and decay of the resonant motion. In figure 3, the details of the 'fast' oscillations are removed from the response plots corresponding to two circular cylinders of radius 0.35 and with centres a distance 0.8 apart. This emphasises the overall shape of the response envelope, which is of primary interest, and allows a straightforward comparison of the results. A reasonable agreement is again found for both types of incident wave.

References

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