Evolution Equation of a Potential Flow with a Free Surface and Moving Solid Boundaries

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1 Introduction

The control of floating bodies on waves is being used increasingly. In the case of ships the aim of the control is usually to reduce its motion to improve its operation. In the case of wave energy converters, the control is even more a core element of their function. They can be seen as devices radiating waves which have to cancel as much as possible the incoming waves. A wave energy converter can thus be considered as a device permitting the control of the motion of the free surface.

Efficient control of any physical system requires a model of its behaviour. The model used can be a theoretical one or a model identified from real measurements. Even in this last case, the mathematical form of the model has to be already known. The purpose of this article is to present the hydrodynamic part of such models. This has the same form for any floating body.

2 System and Bound-Graphs Theories

All deterministic and continuous time models involved in control must be in the form defined by the System and Bound-Graphs Theories. This can summarized as follows:

a) The behaviour of the system[1] can be represented by the evolution equation (EE) of its state variables (SVs), i.e. an equation of the form

$$\frac{de}{dt} = f_1(t, e, u) \tag{1}$$

where t is the time, and e and u are respectively the SVs and the inputs of the system. Input means a variable defined outside of the system, *i.e.* not depending on its SVs. This EE must be canonical, which means that all the SVs in the RHS of (1) must be in its LHS and *vice versa*. Except for electrical circuits, the System Theory does not give rules to determine what are the SVs of a given physical system other than that a complete set of SVs is obtained when a canonical EE can be written with them. The SVs can be chosen freely as long as they match this rule. The function f_1 must be an instantaneous function of the SVs, i.e. a function not depending on time derivatives or integrals of these variables.

b) The outputs v of a system are of the form

$$v = f_2(t, e, u)$$

where f_2 must also be an instantaneous function of the SVs. For real physical systems this function does not depend on the inputs u.

c) In the System Theory inputs and outputs of a system are considered as obvious variables. For real systems these may not be so obvious. This question is solved by the Bound-Graphs Theory. This theory states that an interaction between two systems always consist of an exchange of power and this exchange is always expressed as the product of two variables. It turns out that one of these variables depends only the SVs of one system and the other only the SVs of the other system. This defines the inputs and outputs of a system. The input variable of one system is the output variable of the other.

3 Hydrodynamic Model

The free surface is called F, the hull of the device H, the bottom of the sea B, a boundary of the fluid at infinity B_∞ and the domain filled by the water W. The watertight boundaries are F, H and B. The fluid domain W is bounded by $\partial W = F \cup H \cup B \cup B_\infty$. The reference frame is Galilean with x_1 and x_2 the horizontal coordinates and x_3 the vertical coordinate pointing upwards.

[1] In all this article, the word "system" means "physical system".

3.1 Governing Conditions

3.1.1 Field Conditions

The aim of the model being to represent the motion of the fluid at ocean wave frequencies, the usual hypothesis (irrotationality of the flow, incompressibility of the fluid) leading to a harmonic flow are considered as valid. The Fundamental Principle of Mechanics leads to the Bernouilli equation:

$$(2) \qquad p = -\rho\cdot\psi \qquad \text{with} \qquad \psi = \frac{\partial\varphi}{\partial t} + \frac{1}{2}\cdot(\frac{\partial\varphi}{\partial x_i})^2 - g_i\cdot x_i$$

where p is the fluid pressure, φ the velocity potential and $g_i = [\ 0,\ \ 0,\ \ -g\ ]$ with $g\ (>0)$ the acceleration of gravity. In (2), the reference pressure is $p = 0$ for $\varphi \equiv 0$ and $x_* = 0$. The acceleration of a fluid particle is

$$\gamma_i = \frac{\partial\chi}{\partial x_i} \qquad \text{with} \qquad \chi = \frac{\partial\varphi}{\partial t} + \frac{1}{2}\cdot(\frac{\partial\varphi}{\partial x_i})^2;$$

so χ is the acceleration potential of the flow.

3.1.2 Boundary Conditions

Conditions on Watertight Boundaries This condition is that no fluid particle crosses the boundary. For the velocities this condition is usually expressed by equating the normal component of the fluid velocity to the normal velocity of the boundary surface. This gives the value of $\partial\varphi/\partial n$ on the boundary. For incompressible fluid the condition defining $\partial^2\varphi/\partial t\,\partial n$ has also to be expressed. Such a condition has been written by Tanizawa in [1]. The principle used is to equate the normal component of the acceleration of the fluid to the normal acceleration of the boundary surface. This condition is thus applied to the acceleration potential χ, which is not harmonic, and so indirectly to $\partial\varphi/\partial t$. The determination of these accelerations is a difficult task and the result obtained is curious: added masses do not appear explicitly. To derive systematically the required conditions for the velocity and acceleration fields, we shall follow another approach:

1) All watertight boundaries are defined implicitly, i.e. the coordinates x_{A*} (x_* on A) of the points belonging to the boundary A satisfy a condition of the form $a(t, x_*) = 0$;
2) The watertightness condition is that a fluid particle on the boundary stays on it in a time neighbourhood, which can be expressed by: for $a = 0$,

$$(3.1) \qquad D_t a = 0, \qquad (3.2) \qquad D_t^2 a = 0, \qquad \text{etc.,}$$

where D_t is the material derivative: $D_t = \partial/\partial t + \partial/\partial x_i \cdot \partial\varphi/\partial x_i$. Only the derivatives needed will be used. These conditions are the kinematic conditions to be satisfied by the fluid on the boundary. That for the velocity field is (3.1), which is equivalent to the classical condition for the normal velocities. That for the acceleration field is (3.2). The equivalence or not of this condition with the one of Tanizawa is not established. In any case, (3.2) is the correct condition for the normal accelerations.

These expressions of the kinematic conditions can be used because an implicit definition of the watertight boundaries is always possible: they can be defined by one condition which has to be satisfied on them. In the problem to be modeled, two cases appear:

1) The position of the boundary is given and the pressure is free;
2) The opposite.

The boundaries H and B belong to the case 1). They can be defined by given functions $h = 0$ and $b = 0$. These boundaries giving the same conditions for the fluid, for economy of notation they will be called the boundary R (for rigid): $R = H \cup B$ and $r = h\cdot b$. The boundary F belongs to the case 2). With a given pressure $\overline{p_F}$ applied on it, the condition defining it is $p - \overline{p_F} = 0$.

Condition at Infinity As only the short term transient behaviour of the fluid has to be represented, what happens far away does not need to be taken into consideration. The problem can thus be closed by the assumption that the fluid is at rest at infinity. To obtain simple mathematical proofs of uniqueness, this assumption is expressed by the existence of asymptotic expansions of φ and $\partial\varphi/\partial t$.

3.2 Evolution Equation of the State Variables

The governing conditions are synthesized in the EE of the SVs of the fluid. Harmonicity of φ implies that φ in W can be represented by its values or the values of its normal derivative on ∂W. The boundary R is given so $\partial\varphi/\partial n$ is known on it, the given function r is therefore an input of this EE. With the adopted condition at infinity, the values of φ or $\partial\varphi/\partial n$ on B_∞ do not contribute to φ in W. These considerations applied also to $\partial\varphi/\partial t$. The EE is therefore given only by the free surface conditions. To simplify the expressions, the pressure $\overline{p_F}$ applied on F is taken to be null. These conditions are thus $p = 0$, $D_t p = 0$, $D_t^2 p = 0$, etc. We will see in the following that the first two conditions are necessary and sufficient to write the EE. Their developed expressions are

$$(4)\qquad \begin{cases} (1) & \dfrac{\partial\varphi}{\partial t} + \dfrac{1}{2}\cdot(\dfrac{\partial\varphi}{\partial x_i})^2 - g_i\cdot x_i = 0, \\ (2) & \dfrac{\partial^2\varphi}{\partial t^2} + (2\cdot\dfrac{\partial^2\varphi}{\partial t\,\partial x_i} + \dfrac{1}{2}\cdot\dfrac{\partial}{\partial x_i}((\dfrac{\partial\varphi}{\partial x_j})^2) - g_i)\cdot\dfrac{\partial\varphi}{\partial x_i} = 0. \end{cases}$$

The first condition is necessary to determine the position of F but is not sufficient: a canonical EE cannot be written only with it. We can thus try the first two conditions[2]. They are sufficient if a canonical EE can be written with them. The main question is then: what can be SVs of this EE? We can try the highest time derivatives appearing in each equation. These derivatives are $\partial\varphi/\partial t$ and $\partial^2\varphi/\partial t^2$, so φ_F and $\partial\varphi_F/\partial t$ (φ and $\partial\varphi/\partial t$ on F) can be candidates. Nevertheless, this EE is not canonical since: 1) spacial derivatives of φ and $\partial\varphi/\partial t$ appear in it and: 2) x_{F*}, the position of F, has to be determined. The first point is solved by expressing φ and $\partial\varphi/\partial t$ in W with expressions depending only on φ_F and $\partial\varphi_F/\partial t$. These expressions are obtained by applying the second Green formula to φ or $\partial\varphi/\partial t$ and to the solution $\boldsymbol{g}$ of the problem

$$\begin{cases} \dfrac{\partial^2\boldsymbol{g}(x_*;y_*)}{\partial y_i^2} = -\,\delta(x_*-y_*) & \forall\, x_*\in W, \quad \forall\, y_*\in W, \\ n_i(y_*)\cdot\dfrac{\partial\boldsymbol{g}(x_*;y_*)}{\partial y_i} = 0 & \forall\, x_*\in W, \quad \forall\, y_*\in R, \\ \boldsymbol{g}(x_*;y_*) = 0 & \forall\, x_*\in W, \quad \forall\, y_*\in F, \\ \text{an asymptotic expansion of } \boldsymbol{g} \text{ exists} & \forall\, x_*\in W, \quad \text{horizontal component of } y_*\to\infty \end{cases}$$

where n_* is the outward unitary normal derivative. For φ, the result is

$$(5)\qquad \varphi(x_*) = (-\,\boldsymbol{G}_F(\varphi_F) + \boldsymbol{G}_{Rn}(\frac{\partial r_R}{\partial y_i}\cdot\frac{\partial\varphi_R}{\partial y_i}))(x_*) \quad (x_*\in W,\ y_*\in R)$$

where

$$\boldsymbol{G}_F(\bullet_F)(x_*) = \int_{y_*\in F} ds(y_*)\,(n_i(y_*)\cdot\frac{\partial\boldsymbol{g}(x_*;y_*)}{\partial y_i}\cdot\bullet(y_*))$$

and

$$\boldsymbol{G}_{Rn}(\frac{\partial r_R}{\partial y_i}\cdot\frac{\partial\bullet_R}{\partial y_i})(x_*) = \int_{y_*\in R} ds(y_*)\,(\|\frac{\partial r(y_*)}{\partial y_*}\|^{-1}\cdot\boldsymbol{g}(x_*;y_*)\cdot\frac{\partial r(y_*)}{\partial y_j}\cdot\frac{\partial\bullet(y_*)}{\partial y_j}),$$

"$\bullet$" standing for φ. For infinite depth, the part of $\boldsymbol{G}_{Rn}$ coming from B is null. A term of the form $\partial r_R/\partial x_i\cdot\partial\varphi_R/\partial x_i$ still appears in this representation of φ. This point is solved by using the kinematic condition (3.1) on R, which can be written

$$\frac{\partial r_R}{\partial x_i}\cdot\frac{\partial\varphi_R}{\partial x_i} = -\,\frac{\partial r_R}{\partial t}.$$

This equation carried into (5) gives the representation of φ in W depending only on φ_F and r:

$$(6)\qquad \varphi(x_*) = -(\boldsymbol{G}_F(\varphi_F) + \boldsymbol{G}_{Rn}(\frac{\partial r_R}{\partial t}))(x_*).$$

[2] With a lot of courage, it would be possible to use the higher order conditions, resulting in higher order EEs. This would imply the usage of the corresponding higher order watertightness conditions.

The representation of $\partial\varphi/\partial t$ is slightly more complicated. Application on the second Green formula to $\partial\varphi/\partial t$ and $\boldsymbol{g}$ gives

$$\frac{\partial\varphi}{\partial t}(x_*) = (\,-\,\boldsymbol{G}_F(\frac{\partial\varphi_F}{\partial t}) + \boldsymbol{G}_{R_n}(\frac{\partial r_R}{\partial y_i}\cdot\frac{\partial^2\varphi_R}{\partial t\,\partial y_i}))(x_*) \quad (x_* \in W,\ y_* \in R)$$

where $\boldsymbol{G}_F$ and $\boldsymbol{G}_{R_n}$ keep the same definitions, "•" standing now for $\partial\varphi/\partial t$. A term of the form $\partial r_R/\partial x_i \cdot \partial^2\varphi_R/\partial t\,\partial x_i$ still appears in this representation. This point is solved by using the kinematic condition (3.2) on R, which can be written

$$\frac{\partial r_R}{\partial x_i}\cdot\frac{\partial^2\varphi_R}{\partial t\,\partial x_i} = -\frac{\partial^2 r_R}{\partial t^2} - 2\cdot\frac{\partial^2 r_R}{\partial t\,\partial x_i}\cdot\frac{\partial\varphi_R}{\partial x_i} - \frac{\partial^2 r_R}{\partial x_i\,\partial x_j}\cdot\frac{\partial\varphi_R}{\partial x_i}\cdot\frac{\partial\varphi_R}{\partial x_j} - \frac{\partial r_R}{\partial x_i}\cdot\frac{\partial^2\varphi_R}{\partial x_i\,\partial x_j}\cdot\frac{\partial\varphi_R}{\partial x_j}.$$

If the boundary is motionless, the RHS of this condition is null, *c.f.* [2] § 4.2.1.2.3. If the boundary is moving, gradients of φ still appear in it. This point is solved by carrying into it the representation (6). These representations of φ and $\partial\varphi/\partial t$ carried into (4) give the first part of a canonical EE. All the details of this procedure are given in [2] § 4.2. As a result, added masses appear explicitly in device motion equations. To obtain a complete canonical EE, x_{F*} has also to be determined. In a purely Lagrangian formulation, the EE of this variable is

$$(7.1) \qquad \frac{dx_{F\,i}}{dt} = \frac{\partial\varphi_F}{\partial x_i}, \qquad\qquad (7.2) \qquad \frac{d^2x_{F\,i}}{dt^2} = \frac{\partial\chi_F}{\partial x_i}.$$

Thus x_{F*} could also be considered as a SV of the fluid system, but this would be unsound: physically it is an output (*i.e.* the position of a boundary if the pressure is imposed or *vice versa*). The variables x_{F*} and $\partial\varphi_F/\partial t$ being linked by (4.1), only one of them is independent. Whatever the one chosen as SV, the EEs of both have to be used: the complete canonical EE is composed of (4) and (7). In the present modeling, the chosen SVs are φ_F and $\partial\varphi_F/\partial t$; this leads to more direct expressions.

The complete canonical EE obtained above is called the (Langrangian) $\varphi\varphi_t$ model.

4 Conclusion - Numerical Sea

This model would be interesting as a theoretical model, *i.e.* as a numerical sea[3]. It is an alternative to the Mixed Eulerian-Lagrangian (MEL) method, cf. [3], with its evolution which is to consider $\partial\varphi/\partial t$ as an independent variable. The first main difference with these methods is not so much the use of $\partial\varphi_F/\partial t$ instead of x_{F*} as SV but the fact that only (7.1) is used as EE of x_{F*}; (7.2) is usually forgotten. The other main difference is the kinematic boundary condition used for $\partial\varphi/\partial t$.

Exact free surface models being time varying and nonlinear cannot reasonably be used as mathematical forms to be identified. For that to be possible they can be developed in perturbation[4]. If the zero order solution is time invariant, which is the case for wave energy converters, the resulting model is also time invariant. Even in this case, the identification of just the linearized model would be difficult: its spectrum is continuous. Discrete modes, the trapped modes, may also be present and the combination of the two types of spectrum would not ease this identification.

References

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[2] J.-M. Forestier, *Étude exploratoire de la prédiction en temps réel des mouvements des navires sur la houle*. Ph.D. thesis, École Normale Supérieure, Centre de Mathématiques et Leurs Applications, Cachan, France, Feb. 2005, http://tel.ccsd.cnrs.fr/tel-00011323/.

[3] M. S. Longuet-Higgins and E. D. Cokelet, “The deformation of steep surface waves on water. I. A numerical method of computation,” *Proc. R. Soc. Lond. A* **350**, pp. 1–26, 1976.

[3] "Numerical sea" instead of "Numerical wave tank", the existence of asymptotic expansions of φ and $\partial\varphi/\partial t$ allowing for infinite horizontal dimensions of the fluid domain. These conditions probably act also as absorbing ones.

[4] All the details of the development in perturbation of the $\varphi\varphi_t$ model are given in [2] § 5.2. The implicit definition of the boundaries allows the use of the Distribution Theory which leads straight to the perturbation developments of the boundary conditions *and of the forces applied by the fluid to the boundaries* at any order.