The effect of surface tension on trapped modes in water wave problems

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Introduction

For over half a century there has been considerable interest in the question of under what circumstances the linear water wave problem may admit non-unique mathematical solutions. There have been many results proving the uniqueness of specific classes of systems and until recently it was generally believed that a uniqueness proof for all configurations might be possible to construct. However, examples of non-uniqueness have now been reported for both surface-piercing and submerged bodies (see for example McIver, 1996 and 2000). These non-unique solutions correspond to a trapping of energy around the bodies. To the authors' knowledge, no previous investigation has been made on the existence of trapped modes in free surface water wave problems with surface tension in an unbounded fluid. Its effect shall be considered in a problem where trapped modes are known to exist in its absence.

Formulation of the problem

A two-dimensional Cartesian coordinate system (x, y) is adopted, with the y-axis pointing vertically downwards. The fluid depth is infinite. A time-harmonic velocity potential $\mathrm{Re}\{\phi(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ is assumed, where ω is the angular frequency. With surface tension T, free surface waves have wavenumber k_0 given by the positive real root of $\frac{T}{\rho g}k_0^3 + k_0 = \frac{\omega^2}{g}$, and it is possible to non-dimensionalise the trapped mode problem for $\phi(x, y)$ to the following system of equations:

$$\nabla^2\phi = 0 \quad \text{in the fluid,} \tag{1}$$

$$\frac{\partial\phi}{\partial y} + (1+s)\phi - s\frac{\partial^3\phi}{\partial x^2\partial y} = 0 \quad \text{on the free surface,} \tag{2}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on fixed, rigid boundaries,} \tag{3}$$

$$|\nabla\phi| \to 0 \quad \text{as } x^2 + y^2 \to \infty \;\; (y \geq 0), \tag{4}$$

where the parameter $s = \frac{Tk_0^2}{\rho \mathrm{g}} = \frac{\omega^2}{gk_0} - 1$ has been introduced. For fixed ω, fluid density ρ and g, s is a measure of the surface tension present.

Trapped modes around submerged bodies

An example of a submerged structure that supports trapped modes is given by McIver (2000). This is constructed by placing a horizontal and vertical dipole pairing at (a, h) and a symmetric combination at $(-a, h)$. The constant a is chosen to be $\frac{\pi}{4}$, whereas h is allowed to vary. The exact velocity potential is given by

$$\begin{aligned}\phi(x, y, a, h) = {} & \sin a(\phi_v(x, y, a, h) + \phi_v(x, y, -a, h)) \\ & - \cos a(\phi_h(x, y, a, h) - \phi_h(x, y, -a, h)),\end{aligned}$$

where ϕ_h (ϕ_v) denotes the potential of a horizontal (vertical) dipole:

$$\phi_h(x,y,\xi,\eta) = \frac{1}{1+s}\left(\frac{x-\xi}{(x-\xi)^2+(y-\eta)^2} + \frac{x-\xi}{(x-\xi)^2+(y+\eta)^2}\right)$$
$$+2\,\mathrm{Im}\, \int_0^\infty \frac{\mathrm{e}^{-m(y+\eta)+\mathrm{i}m(x-\xi)}}{sm^3+m-1-s}\mathrm{d}m, \tag{5}$$
$$\phi_v(x,y,\xi,\eta) = \frac{1}{1+s}\left(\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2} - \frac{y+\eta}{(x-\xi)^2+(y+\eta)^2}\right)$$
$$-2\,\mathrm{Re}\, \int_0^\infty \frac{\mathrm{e}^{-m(y+\eta)+\mathrm{i}m(x-\xi)}}{sm^3+m-1-s}\mathrm{d}m. \tag{6}$$

For each $h > 0$, there exist four saddle-point stagnation points — two in $x > 0$ and two in $x < 0$. By considering the variation of streamfunction values at the two stagnation points in $x > 0$, it is argued that there must be a positive value of h, h_0 say, at which the streamfunction takes the same numerical value ψ_0 at these two points. These stagnation points can then be shown to be connected by a closed streamline. Symmetry shows that the stagnation points in $x < 0$ are also connected by a closed mirror-image streamline. These two closed streamlines can be interpreted as the boundaries of two submerged bodies. The argument principle is used to show that the stagnation points lie in $y > 0$, and a free surface plot verifies that $\psi(x, 0) = \psi_0$ on the 'other' branch of the streamline (i.e. the one not being interpreted as a body boundary — see figure 1). When

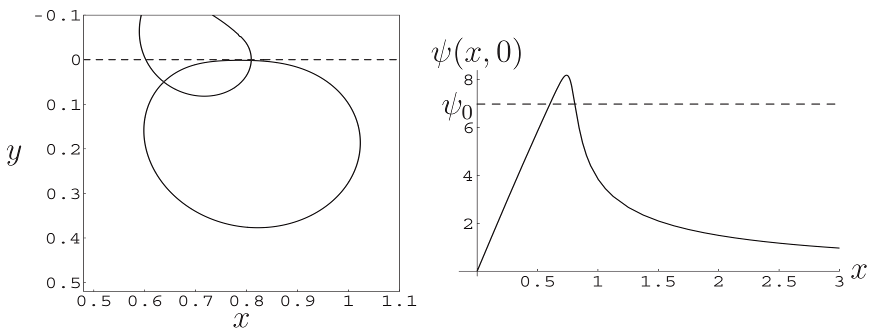

Figure 1: The submerged body in $x > 0$ that supports trapped modes, and a plot of the streamfunction values on the free surface, both presented in McIver (2000).

surface tension is included, the same combination of dipoles satisfies (1), (2) and (4), and it can be shown that the corresponding complex potential $w = \phi + \mathrm{i}\psi$ is given by

$$w(z) = \frac{-2z_0\mathrm{e}^{-\mathrm{i}a}}{(1+s)(z^2-z_0^2)} - \frac{2\bar{z}_0\mathrm{e}^{\mathrm{i}a}}{(1+s)(z^2-\bar{z}_0^2)}$$
$$-\frac{2\mathrm{i}\mathrm{e}^{-\mathrm{i}a}}{3s+1}\left(g(z+z_0)+B_+p_+(z+z_0)+B_-p_-(z+z_0)\right)$$
$$+\frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}a}}{3s+1}\left(g(z-\bar{z}_0)+B_+p_+(z-\bar{z}_0)+B_-p_-(z-\bar{z}_0)\right), \tag{7}$$

where $z = x + \mathrm{i}y$, $z_0 = a + \mathrm{i}h$ and $B_\pm = -\frac{1}{2} \pm \frac{3\mathrm{i}}{2}\left(3 + \frac{4}{s}\right)^{-\frac{1}{2}}$. The functions $g(z)$, $p_\pm(z)$ arise from the partial fraction decomposition as in (5). They are defined in terms of the exponential integral $\mathrm{E}_1(z)$, in such a way as to ensure that the branch cut of each function lies on $x = 0$, $y < 0$ (note that the functions $p_\pm(z)$ are not needed for $s = 0$ as there are then no complex zeros of $sm^3 + m - 1 - s$).

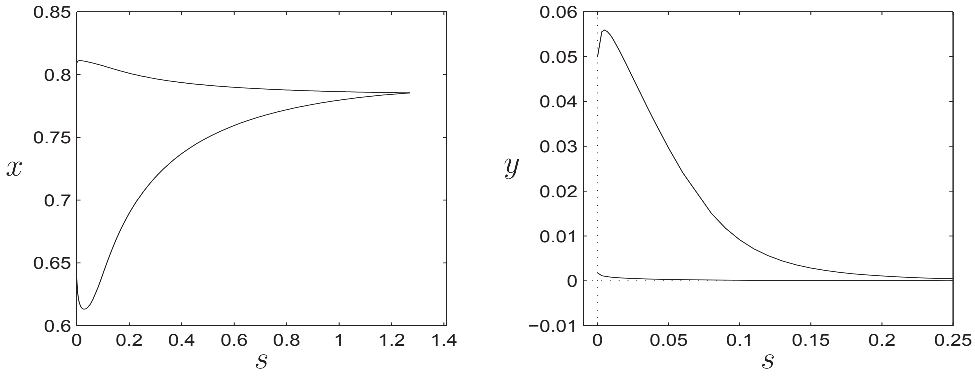

Figure 2: The x and y location of the stagnation points as a function of s $(a = \frac{\pi}{4})$.

With $a = \frac{\pi}{4}$, contour plots of the bodies when $s \neq 0$ closely resemble the one shown in figure 1. However, as surface tension increases the stagnation points in $x > 0$ converge upon either side of $x = \frac{\pi}{4}$ and move closer to the free surface, as can be seen from the second graph in figure 2. The dipole submergence h_0 also becomes very small. In addition, as s increases it becomes increasingly difficult to accurately calculate the position of the stagnation points. Indeed, by around $s = 1.275$ the results found (using Newton's method) can no longer be taken to be accurate. The reason for this is that for each value of a, there is a critical value of s, s_0 say, above which the trapped mode no longer persists. This can be shown using asymptotic expansions. Assume that $h_0 = \epsilon \ll 1$ and seek an expansion for s of the form $s = s_0 - (\alpha_1\epsilon + \alpha_2\epsilon\log\epsilon + \alpha_3\epsilon^2 + \alpha_4\epsilon^2\log\epsilon + \dots)$, where α_i are constants to be found and the $\log\epsilon$ terms are needed to ensure the convergence of certain integrals. It can be shown that the stagnation points in $x > 0$ are located at $a + \epsilon(-\tan a \pm \sec a) \pm \epsilon^3 q_\pm$, where $q_\pm = \pm\sec^2 a\left(\frac{\cos a}{2a^2} - c_0(s_0)\right)(\sec a \mp \tan a)^2$, with

$$c_0(s_0) = -2(1 + s_0)\mathrm{e}^{-\mathrm{i}a}\int_0^\infty \frac{m(\mathrm{e}^{2\mathrm{i}a} - \mathrm{e}^{2\mathrm{i}ma})}{s_0m^3 + m - 1 - s_0}\mathrm{d}m. \tag{8}$$

It can be seen that if the imaginary part of $c_0(s_0)$ is non-zero, then the stagnation points lie on opposite sides of the free surface, and the body that supports trapped modes ceases to be submerged. It follows that to retain the possibility of a submerged body, $\mathrm{Im}(c_0(s_0)) = 0$. This condition allows us to find s_0 for a given value of a. Figure 3 shows how s_0 varies as a. In particular, when $a = \frac{\pi}{4}$, $s_0 \approx 1.277$. This explains the difficulties encountered when trying to obtain numerical results for $a = \frac{\pi}{4}$, $s > 1.27$.

The next order terms in the expansion give the depths of the stagnation points below the free surface, and a condition on ϵ that ensures the stream-function takes the same value at these points. Omitting the details, it is found that for a given dipole submergence ϵ, the approximate value of s that ensures $\psi = \psi_0$ at the two stagnation points for $a = \frac{\pi}{4}$ is, up to $\mathcal{O}(\epsilon)$, $s =$

$1.276997 + \epsilon(3.24 + 23.2 \log \epsilon)$, provided ϵ is small. This last equation agrees well with the numerical results found when ϵ is less than approximately 5×10^{-4}, which corresponds to the range $1.19 < s < s_0$. For these values of ϵ the error in the approximation is at most 0.37%.

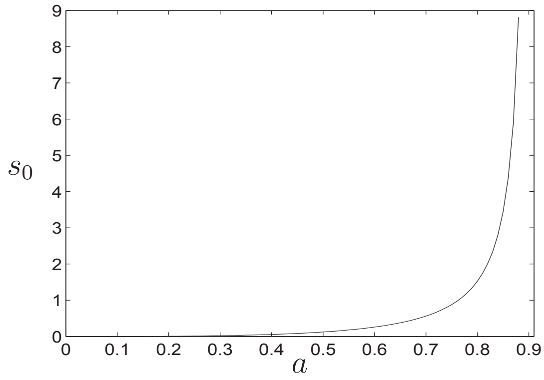

Figure 3: Graph showing how the critical surface tension value varies with a.

Examining expression (8) from a different perspective, it can be supposed that for every critical value s_0 there is a minimum dipole separation, $a_{\min}$ say, below which there are no submerged obstacles that support trapped modes. Of course, $a_{\min}$ is related to the minimum allowable body separation. For a given s_0, $a_{\min}$ is given by figure 3. It might be expected that there is also a maximum dipole separation $a_{\max}$. Experimentation on the variation of h_0 as a function of a suggests that for all values of s, $a_{\max} = \frac{\pi}{2}$. The most deeply submerged bodies are produced when a lies in the vicinity of $\frac{\pi}{4}$ and the global maximum value of h_0 that supports submerged trapped modes is 0.0597177, attained when $s = 0.013$, $a = 0.7652$. When there is no surface tension present, the maximum obtainable dipole depth is 0.0547949, which occurs when $a = 0.7756$. Figure 3 also suggests that when a approaches a value close to 0.9, s_0 diverges to infinity. By deforming the contour in (8) and rearranging, it can be shown that indeed $s_0 \to \infty$ when $a \to 0.901924\ldots$. Therefore, beyond this value of a there is no critical value for the surface tension, i.e. submerged bodies which support a trapped mode exist for all values of s.

Conclusion

It has been shown that the exclusion of surface tension from the linearized water wave problem is not always justified, as its inclusion in the particular submerged-body example above changes the qualitative (i.e. topological) nature of the streamline pattern. This is not just a hypothetical result — the breakdown of the existence of localized solutions about these submerged bodies occurs at physically realistic wavelengths, provided that the parameter a is chosen appropriately.

References

McIver, M. 1996 ‘An example of non-uniqueness in the two-dimensional linear water wave problem.’ *J. Fluid Mech.* **315**, 257–266

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