Reciprocity Relations of Waves Generated by an Antisymmetric Floating Body

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1. Introduction

A large number of theoretical studies based on the potential flow have been made so far to understand wave-making characteristics of a floating body in waves. As a result, a simple theoretical formula for perfect wave reflection/transmission is known for a symmetric floating body, in addition to hydrodynamic properties such as the Haskind-Newman relation, wave-energy conservation, and a relation between the scattered and radiated waves [1] [2].

On the other hand, for an antisymmetric floating body, it was proven by Bessho [3] for the diffraction problem that (1) both amplitude and phase of the transmission wave are the same and (2) the amplitude of the reflection wave is the same, irrespective of the incoming direction of an incident wave. It was also shown by Murashige & Kinoshita [4] that the wave energy of the symmetric and antisymmetric components of body-generated waves are not the same for an antisymmetric body. However the proof by Murashige & Kinoshita was only for the diffraction problem and no explicit formula for the energy splitting was provided. It is well known for a symmetric body that the energy of symmetric wave and of antisymmetric wave are equal (which is referred to as the equally wave-energy splitting law), which holds for not only the diffraction problem but also the case of body motions being free [5].

Recently experiments were conducted for an antisymmetric body in shallow water in connection with development of a floating pier; which obviously shows that the characteristics of (1) and (2) proved by Bessho for a fixed antisymmetric body are also true even when the body is oscillating in an incident wave. The present paper provides first a theoretical proof for these characteristics of the wave reflection and transmission by an antisymmetric 2-D floating body that is oscillating in regular progressive waves. The proof is an extension of Newman [2] based on Green's theorem. It is also shown that the energy of symmetric (antisymmetric) wave component when the incident wave is incoming from the right is equal to the energy of antisymmetric (symmetric) wave component when the incident wave is incoming from the left.

Not only a theoretical study but also numerical confirmation is performed with an antisymmetric Lewis-form body. It is shown that the wave exciting forces and resultant wave-induced motions vary depending on the direction of incident wave, and yet the transmission wave is completely the same, independent of the direction of incident wave, and this is also the case for the amplitude of the reflection wave.

2. Definition of Problem

Assuming the fluid to be incompressible and inviscid with irrotational motion, we introduce the velocity potential and consider the flow around a 2-D floating body in regular incident waves. The wave-induced motion of a body and associated fluid motion are assumed to be linear in the incident-wave amplitude and harmonic in time with circular frequency ω of the incident wave. In what follows, all oscillatory quantities will be expressed in complex form, with the time dependence $e^{i\omega t}$ understood.

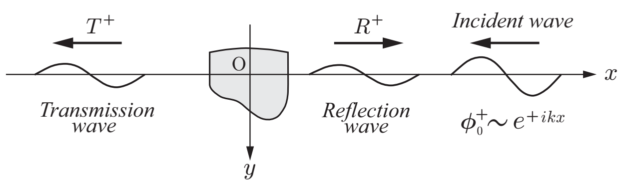

Fig. 1 Reflection and transmission waves for an incident wave incoming from the positive x-axis.

In order to treat the problem in general, we consider an antisymmetric body; in this case, depending on the incoming direction of incident wave, the flow field around a body may be different. Accordingly, as shown in Fig. 1, let us first consider the case where the incident wave is incoming from the positive x-axis (which may be referred to as the 'positive' incident wave) and write the resulting velocity potential in the form

$$\phi^+(x,y) = \frac{g\zeta_a}{i\omega}\Big\{ \phi_D^+(x,y) - K X_j^+ \phi_j(x,y) \Big\}$$
$$\equiv \frac{g\zeta_a}{i\omega}\, \varphi^+(x,y), \tag{1}$$

$$\phi_D^+ = \phi_0^+ + \phi_4^+, \quad \phi_0^+ = \frac{\cosh k(y-h)}{\cosh kh}\, e^{ikx}, \tag{2}$$

$$K = \frac{\omega^2}{g} = k \tanh kh, \tag{3}$$

where ζ_a is the amplitude of incident wave and g is the acceleration of gravity. ϕ_D^+ is the diffraction potential which is the sum of the incident-wave potential ϕ_0^+ and the scattering potential ϕ_4^+. The water depth is assumed to be finite and constant h in this paper, and the wavenumber k of a progressive wave satisfies the dispersion relation given by (3). X_j^+ denotes the complex amplitude of the body motion in the j-th mode ($j = 1$ for sway, $j = 2$ for heave, and $j = 3$ for roll), and ϕ_j is the radiation potential with unit velocity in the j-th direction (which is independent of the incident wave and hence no superscript is attached). The summation sign with respect to j is deleted throughout the paper with the convention that any term of an equation containing the same index twice should be summed over that index.

The asymptotic expression of the normalized velocity potential at $x \to \pm\infty$ and $y = 0$ may be expressed as follows:

$$\varphi^+(x,0) \sim \begin{cases} e^{ikx} + R_F^+ e^{-ikx} & \text{as } x \to +\infty \\ T_F^+ e^{ikx} & \text{as } x \to -\infty \end{cases} \tag{4}$$

where

$$\left.\begin{aligned} R_F^+ &= R_D^+ - iKX_j^+ H_j^+ , \quad R_D^+ = iH_4^+ \\ T_F^+ &= T_D^+ - iKX_j^+ H_j^- , \quad T_D^+ = 1 + iH_4^- \end{aligned}\right\} \tag{5}$$

R^+ and T^+ are defined as the coefficients of reflection and transmission waves, respectively. Suffix D to these coefficients indicates the quantities for the diffraction problem; likewise suffix F indicates the quantities for the case where a body is freely oscillating in an incident wave. These coefficients and the amplitudes of wave-induced motions of a body are nondimensionalized using the incident-wave amplitude. $H_4^\pm$ and $H_j^\pm$ $(j = 1 \sim 3)$ denote the Kochin functions associated with the far-field scattered and radiated waves, respectively.

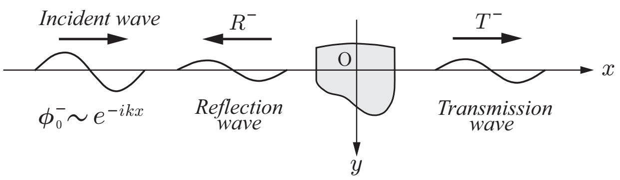

Fig. 2 Reflection and transmission waves for an incident wave incoming from the negative x-axis.

In a similar manner, we consider the case of Fig. 2 where the incident wave is incoming from the negative x-axis (which may be referred to as the 'negative' incident wave) and write the corresponding velocity potential in the form

$$\begin{aligned} \phi^-(x,y) &= \frac{g\zeta_a}{i\omega}\Big\{ \phi_D^-(x,y) - KX_j^- \phi_j(x,y) \Big\} \\ &\equiv \frac{g\zeta_a}{i\omega}\,\varphi^-(x,y), \end{aligned} \tag{6}$$

$$\phi_D^- = \phi_0^- + \phi_4^- , \quad \phi_0^- = \frac{\cosh k(y-h)}{\cosh kh}\, e^{-ikx}. \tag{7}$$

Here the minus sign as superscript implies the quantities in the 'negative' incident wave.

The asymptotic expression of the normalized velocity potential at $x \to \pm\infty$ and $y = 0$ takes the following form:

$$\varphi^-(x,0) \sim \begin{cases} T_F^- e^{-ikx} & \text{as } x \to +\infty \\ e^{-ikx} + R_F^- e^{ikx} & \text{as } x \to -\infty \end{cases} \tag{8}$$

where

$$\left.\begin{aligned} R_F^- &= R_D^- - iKX_j^- H_j^- , \quad R_D^- = ih_4^- \\ T_F^- &= T_D^- - iKX_j^- H_j^+ , \quad T_D^- = 1 + ih_4^+ \end{aligned}\right\} \tag{9}$$

Here the scattered wave is different from that in the former case and thus the associated Kochin function is expressed as $h_4^\pm$. (Note that $h_4^\pm = H_4^\mp$ for a body with port-and-starboard symmetry.)

In order to compute the reflection and transmission waves, the diffraction and radiation potentials must be determined first and then the Kochin functions and wave-induced motions must be evaluated, which will be described next.

3. Numerical Solution Method

The diffraction ($\psi_j = \phi_D^\pm$) and radiation ($\psi_j = \phi_j$) potentials are determined directly by solving an integral equation for the velocity potential ψ_j, expressed in the form

$$\begin{aligned} &C(\mathrm{P})\psi_j(\mathrm{P}) + \int_{S_H} \psi_j(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} G(\mathrm{P};\mathrm{Q})\, d\ell(\mathrm{Q}) \\ &\quad = \begin{cases} \phi_0^\pm(\mathrm{P}) \quad (j = D) \\ \displaystyle\int_{S_H} n_j(\mathrm{Q}) G(\mathrm{P};\mathrm{Q})\, d\ell(\mathrm{Q}) \quad (j = 1 \sim 3) \end{cases} \end{aligned} \tag{10}$$

Here S_H denotes the body surface below $y = 0$, on which the body boundary conditions

$$\frac{\partial \phi_D^\pm}{\partial n} = 0, \quad \frac{\partial \phi_j}{\partial n} = n_j \tag{11}$$

are satisfied, where n_j denotes the j-th component ($n_1 = n_x$, $n_2 = n_y$, and $n_3 = xn_2 - yn_1$) of the normal vector, which is defined as positive outward from the body surface. $\mathrm{P} = (x,y)$ and $\mathrm{Q} = (\xi,\eta)$ denote the field and integration points, respectively, located on the body surface and $C(\mathrm{P})$ denotes the solid angle. $G(\mathrm{P};\mathrm{Q})$ represents the free-surface Green function in water of constant finite depth.

The integral equation (10) is solved by the so-called constant-panel and collocation method with a remedy for getting rid of the irregular frequencies. The free-surface Green function is evaluated using a power-series expression for relatively large values of $|x-\xi|$ and the integral expressions for other values of $|x-\xi|$.

Once the velocity potentials on the body surface are determined, it is straightforward to compute the hydrodynamic forces. With the convention that all quantities are written in nondimensional form, the hydrodynamic forces in the diffraction and radiation problems are expressed in the form

$$\left.\begin{aligned} E_j^\pm &= \int_{S_H} \phi_D^\pm n_j \, d\ell, \\ f_{jk} &= -\int_{S_H} \phi_k n_j \, d\ell = A_{jk} - iB_{jk}, \end{aligned}\right\} \tag{12}$$

where $E_j^\pm$ is the wave-exciting force in the j-th direction, and A_{jk} and B_{jk} are the added-mass and damping coefficients, respectively, in the j-th direction due to the k-th mode of motion.

In terms of these forces, the equations of body motion with respect to the origin of the coordinate system can be expressed in a matrix form as

$$\Big[-K\big(M_{jk} + f_{jk}\big) + C_{jk} \Big] X_k^\pm = E_j^\pm , \tag{13}$$

for $j = 1 \sim 3$, where M_{jk} denotes the mass matrix and its nonzero values are in the diagonal ($j = k$), which are the body mass (m) for $j = 1$ and 2 and the moment of inertia for $j = 3$, and also $M_{13} = M_{31} = -my_G$ and $M_{23} = M_{32} = mx_G$ for off-diagonals, where (x_G, y_G) is the position of the center of gravity, generally unequal to the origin of the coordinate system due to asymmetry of a body. C_{jk} denotes the restoring force coefficients due to the static pressure. It should be noted that both M_{jk} and C_{jk} are real quantities and the symmetry relation of $M_{jk} = M_{kj}$ and $C_{jk} = C_{kj}$ holds as is the same for the added-mass and damping coefficients.

4. Hydrodynamic Relations

In order to derive some important reciprocity and energy-conservation relations for the reflection and transmission waves, Green's theorem can be applied to two different velocity potentials. The idea was originally proposed by Newman [2], and the basis equation for derivation of various hydrodynamic relations is of the following form:

$$\int_{S_H}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)d\ell=\int_0^h dy\left[\phi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\phi}{\partial x}\right]_{-\infty}^{+\infty}$$
$$=\frac{1}{2kC_0}\left[\left(\phi\frac{\partial\psi}{\partial x}-\psi\frac{\partial\phi}{\partial x}\right)_{y=0}\right]_{-\infty}^{+\infty} \qquad (14)$$

where
$$C_0=\frac{k}{K+h(k^2-K^2)}. \qquad (15)$$

Here both potentials ϕ and ψ must satisfy the same boundary conditions on the free surface and water bottom, but not necessarily the same boundary conditions on the body surface (S_H) and the radiation surface at $x=\pm\infty$. The square brackets with superscript $+\infty$ and subscript $-\infty$ means the difference between the quantities in the brackets evaluated at $x=+\infty$ and $x=-\infty$.

As the first application of (14), we consider φ^+ for ϕ and φ^- for ψ. In this case, the left-hand side of (14), denoted by $\mathcal{L}$, may be evaluated as follows:

$$\begin{aligned}\mathcal{L}&=\int_{S_H}\Big\{\big(\phi_D^+-KX_j^+\phi_j\big)\big(-KX_k^-n_k\big)\\&\qquad-\big(\phi_D^--KX_k^-\phi_k\big)\big(-KX_j^+n_j\big)\Big\}d\ell\\&=-KX_k^-\big(E_k^++KX_j^+f_{kj}\big)\\&\quad+KX_j^+\big(E_j^-+KX_k^-f_{jk}\big)\\&=-KX_k^-X_j^+\big(-KM_{kj}+C_{kj}\big)\\&\quad+KX_j^+X_k^-\big(-KM_{jk}+C_{jk}\big)=0,\end{aligned} \qquad (16)$$

where the body-boundary condition (11), the hydrodynamic forces (12), the equations of body motion (13), and the symmetry relations of $M_{jk}=M_{kj}$ and $C_{jk}=C_{kj}$ have been used.

The right-hand side of (14), denoted by $\mathcal{R}$, may be evaluated by using (4) and (8), and the result becomes

$$\mathcal{R}=\frac{i}{C_0}\big(T_F^+-T_F^-\big). \qquad (17)$$

Therefore $\mathcal{L}=\mathcal{R}$ gives the first important relation:

$$T_F^+=T_F^-. \qquad (18)$$

This means that the coefficient of transmission wave past an antisymmetric body is independent of the incoming direction of an incident wave and must be the same in both amplitude and phase. To be emphasized here is that this relation holds even when the body motions are free to oscillate in response to the incident wave.

Next we consider $\overline{\varphi^+}$ (which is the complex conjugate and physically the reverse-time velocity potential) for ϕ and φ^- for ψ. In this case, in the same way as in obtaining (16), the left-hand side of (14) becomes

$$\begin{aligned}\mathcal{L}&=\int_{S_H}\Big\{\big(\overline{\phi_D^+}-K\overline{X_j^+\phi_j}\big)\big(-KX_k^-n_k\big)\\&\qquad-\big(\phi_D^--KX_k^-\phi_k\big)\big(-K\overline{X_j^+}n_j\big)\Big\}d\ell\\&=-KX_k^-\big(\overline{E_k^+}+K\overline{X_j^+f_{kj}}\big)\\&\quad+K\overline{X_j^+}\big(E_j^-+KX_k^-f_{jk}\big)\\&=-KX_k^-\overline{X_j^+}\big(-KM_{kj}+C_{kj}\big)\\&\quad+K\overline{X_j^+}X_k^-\big(-KM_{jk}+C_{jk}\big)=0,\end{aligned} \qquad (19)$$

where the last transformation has been performed in terms of a fact that M_{jk} and C_{jk} are real quantities in addition to the symmetry relations of $M_{jk}=M_{kj}$ and $C_{jk}=C_{kj}$.

On the other hand, using (4) and (8), the right-hand side of (14) takes the form

$$\mathcal{R}=-\frac{i}{C_0}\big(R_F^-\overline{T_F^+}+\overline{R_F^+}T_F^-\big). \qquad (20)$$

Therefore $\mathcal{L}=\mathcal{R}$ gives the following relation:

$$R_F^-\overline{T_F^+}+\overline{R_F^+}T_F^-=0. \qquad (21)$$

Substituting the relation $T_F^+=T_F^-$ given by (18) into (21), we can obtain another important relation for the reflection wave:

$$\big|R_F^+\big|=\big|R_F^-\big|. \qquad (22)$$

Namely, the amplitude of the reflection wave by an antisymmetric body which is oscillating freely in the incident wave must also be the same, irrespective of the incoming direction of incident wave.

It is obvious from the above proof that the relations of (18) and (22) hold for the diffraction problem. In fact, these relations for the diffraction problem were proved for the first time by Bessho [3] using the idea of the reverse-time velocity potential.

By taking φ^+ for ϕ and $\overline{\varphi^+}$ for ψ (or similarly φ^- for ϕ and $\overline{\varphi^-}$ for ψ) and evaluating (14) in the same manner, we can easily prove that

$$\big|R_F^+\big|^2+\big|T_F^+\big|^2=\big|R_F^-\big|^2+\big|T_F^-\big|^2=1. \qquad (23)$$

This is known as the relation of energy conservation for the case where a body is oscillating freely in regular waves.

In passing, let us consider the followings:

$$\left.\begin{aligned}\big|R^+\pm T^+\big|^2&=\big|R^+\big|^2+\big|T^+\big|^2\pm\big(R^+\overline{T^+}+\overline{R^+}T^+\big)\\\big|R^-\pm T^-\big|^2&=\big|R^-\big|^2+\big|T^-\big|^2\pm\big(R^-\overline{T^-}+\overline{R^-}T^-\big)\end{aligned}\right\} \qquad (24)$$

In terms of (18) and (21), we can see that

$$\overline{R^+}T^+=\overline{R^+}T^-=-R^-\overline{T^+}=-R^-\overline{T^-}. \qquad (25)$$

Accordingly, combining these results, it follows that

$$\big|R^+\pm T^+\big|=\big|R^-\mp T^-\big|. \qquad (26)$$

We note that $(R+T)/2$ and $(R-T)/2$ give the symmetric and antisymmetric wave components, respectively, with respect to $x=0$ and the amplitude is related to the energy of a progressive wave. Therefore, (26) indicates that the energy of symmetric (antisymmetric) wave component when the incident wave is incoming from the right is equal to the energy of antisymmetric (symmetric) wave component when the incident wave is incoming from the left. This new finding can be regarded as the wave-energy splitting law for an antisymmetric body.

For bodies with horizontal symmetry, $R\overline{T} + \overline{R}T = 0$ from (21) and $|R|^2 + |T|^2 = 1$ from (23), and hence

$$\left|R + T\right| = \left|R - T\right| = 1. \tag{27}$$

This relation was established by Kato et al. [5] and Newman [2], implying that the energy of the symmetric wave and of the antisymmetric wave are equal.

5. Numerical Results

Numerical computations were preformed for an antisymmetric body shown in Fig. 3; each of the right and left section shapes is of the Lewis form that can be represented by a conformal mapping in terms of the half breadth-to-draft ratio $H_0 = b/d$ and the sectional area ratio $\sigma = S/bd$. As shown in Fig. 3, the right-hand shape ($x > 0$) is for $H_0 = 1.0$ and $\sigma = 0.95$ and the left-hand shape ($x < 0$) is for $H_0 = 1.0$ and $\sigma = 0.50$.

The position of the center of buoyancy (x_B, y_B) was numerically computed from the section shape and the center of gravity (x_G, y_G) is assumed to be $x_G = x_B$ and $y_G = 0.15d$. The gyrational radius in roll κ_{zz} and the water depth h are taken equal to $\kappa_{zz} = 0.4b$ and $h = 4.0d$, respectively. Numerical accuracy was confirmed to be excellent (virtually perfect) through checking the energy conservation, the Haskind-Newman relation, and so on.

Due to the lack of space, only the final result of the transmission and reflection coefficients is presented in Fig. 4 for the case where all modes of the body motion are free. As theoretically proven, the transmission wave in the 'positive' incident wave and in the 'negative' incident wave (both are shown by broken line) are equal in not only amplitude but also phase. On the other hand, only the amplitude is the same in the reflection wave and obviously the phase is different depending on the incoming direction of incident wave.

6. Conclusions

The present paper provided a theoretical proof for the 'reciprocity' relations with reference to the transmission and reflection waves by an antisymmetric body which is freely oscillating in regular waves. It was shown that, irrespective of the incoming direction of incident wave, the transmission wave (both amplitude and phase) is the same, and the amplitude of reflection wave is also the same independent of whether the incident wave is incoming from the right or left. In addition, the wave-energy splitting law for an antisymmetric body was also derived; this law indicates that the energy of symmetric (antisymmetric) wave when the incident wave is incoming from the right is equal to that of antisymmetric (symmetric) wave when the incident wave is incoming from the left.

These features were verified by numerical computations performed for an antisymmetric Lewis-form body, not only for the case where all modes of body motion are free but also for the case where only the heave motion is free. For both cases, numerical satisfaction of the 'reciprocity' relations and the wave-energy splitting law theoretically proven was virtually perfect.

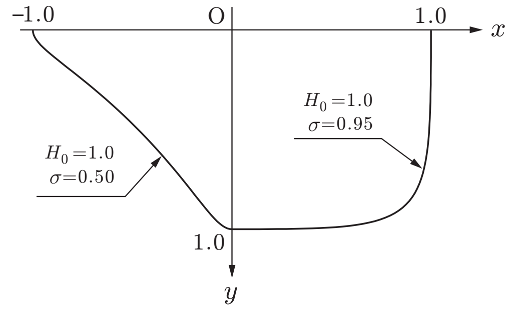

Fig. 3 Section shape of the Lewis-form body used for numerical computations

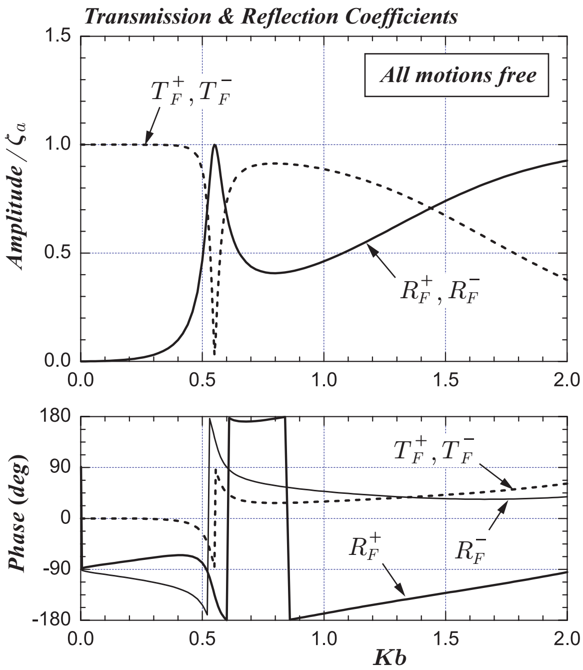

Fig. 4 Transmission and reflection waves by the body shown in Fig. 3 (for the case of all body motions free)

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