# Nearfield and farfield boundary-integral representations of free-surface flows

Francis Noblesse[1](francis.noblesse@navy.mil), Chi Yang[2,3], Rommel Espinosa[2]
[1] NSWCCD, 9500 MacArthur Blvd, West Bethesda, MD 20817, USA
[2] College of Science, George Mason University, Fairfax, VA 22030, USA (cyang@gmu.edu)
[3] Shanghai Jiao Tong University, Shanghai, China

## 1. Green's classical potential representation

Consider a finite 3D region $\mathcal{D}$ bounded by a closed surface $\Sigma$. The divergence theorem applied to the function $\phi\nabla G - G\,\nabla\phi$ yields the classical Green identity

$$\int_{\mathcal{D}} d\mathcal{V}\,(\phi\nabla^2 G - G\,\nabla^2\phi) = \int_{\Sigma} d\mathcal{A}\,(G\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla G) \tag{1}$$

where $d\mathcal{V}$ and $d\mathcal{A}$ stand for differential elements of volume or area of the region $\mathcal{D}$ or the boundary surface $\Sigma$, and $\mathbf{n}$ is a unit vector that is normal to $\Sigma$ and points inside $\mathcal{D}$. For a function $\phi \equiv \phi(\mathbf{x})$ that satisfies the Laplace equation $\nabla^2\phi = 0$ within $\mathcal{D}$, and a Green function $G \equiv G(\mathbf{x}\,;\widetilde{\mathbf{x}})$ that satisfies the Poisson equation $\nabla^2 G = \delta(x-\widetilde{x})\,\delta(y-\widetilde{y})\,\delta(z-\widetilde{z})$ in $\mathcal{D}$, or in a larger region that includes $\mathcal{D}$, (1) yields Green's classical boundary-integral representation

$$\widetilde{C}\,\widetilde{\phi} = \int_{\Sigma} d\mathcal{A}\,(G\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla G) \tag{2a}$$

$$\text{with}\quad \widetilde{C} = \int_{\mathcal{D}} d\mathcal{V}\,\delta(x-\widetilde{x})\,\delta(y-\widetilde{y})\,\delta(z-\widetilde{z}) = \left\{\begin{matrix} 1 \\ 1/2 \\ 0 \end{matrix}\right\} \text{ if } \widetilde{\mathbf{x}} \text{ lies } \left\{\begin{matrix} \text{inside } \mathcal{D} \\ \text{on } \Sigma \\ \text{outside } \mathcal{D} \end{matrix}\right\} \tag{2b}$$

Here and below, $\mathbf{x} = (x\,,y\,,z)$ and $\phi$ are nondimensional in terms of a reference length $L$ and velocity $U$, i.e. one has $\mathbf{x} = \mathbf{X}/L$ and $\phi = \Phi/(UL)$. In (2b), the value $\widetilde{C} = 1/2$ at a point $\widetilde{\mathbf{x}}$ of the boundary surface $\Sigma$ assumes that $\Sigma$ is smooth at $\widetilde{\mathbf{x}}$. Green's representation (2) defines the potential $\widetilde{\phi} \equiv \phi(\widetilde{\mathbf{x}})$ at a flow-field point $\widetilde{\mathbf{x}}$ in terms of boundary distributions of sources (with strength $\mathbf{n}\cdot\nabla\phi$) and normal dipoles (strength $\phi$), and involves a Green function $G$ and the first derivatives of $G$. In (2) and below, $\widetilde{\mathbf{x}}$ stands for a flow-field point, i.e. a point inside a 3D flow region $\mathcal{D}$, and $\mathbf{x}$ is a point of the boundary surface $\Sigma$ of the flow region, i.e. a boundary point.

The general solution of the Poisson equation $\nabla^2 G = \delta(x-\widetilde{x})\,\delta(y-\widetilde{y})\,\delta(z-\widetilde{z})$ is given by

$$4\pi G = -1/r + H = S + H \quad\text{with}\quad r = \sqrt{(x-\widetilde{x})^2 + (y-\widetilde{y})^2 + (z-\widetilde{z})^2} \tag{3}$$

$r$ is the distance between $\mathbf{x} = (x\,,y\,,z)$ and $\widetilde{\mathbf{x}} = (\widetilde{x},\widetilde{y},\widetilde{z})$, and $H(\mathbf{x}\,;\widetilde{\mathbf{x}})$ stands for a function that is harmonic within the flow region $\mathcal{D}$ (or a larger region that includes $\mathcal{D}$). Thus, the singular component $S$ and the harmonic component $H$ in (3) satisfy

$$\nabla^2 S = 4\pi\,\delta(x-\widetilde{x})\,\delta(y-\widetilde{y})\,\delta(z-\widetilde{z}) \quad\text{and}\quad \nabla^2 H = 0$$

Application of Green's identity (1) to the potential $\phi$ and the functions $S$ or $H$ yield

$$4\pi\,\widetilde{C}\,\widetilde{\phi} = \int_{\Sigma} d\mathcal{A}\,(S\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla S) \qquad 0 = \int_{\Sigma} d\mathcal{A}\,(H\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla H) \tag{4}$$

## 2. Application to free-surface flows in deep water

The boundary surface $\Sigma$ and the Green function $G$ in Green's relations (2) and (4) are generic. These generic relations are now applied to free-surface flows about ships or offshore structures in deep water. The closed boundary surface $\Sigma$ in (2a) consists of $\Sigma = \Sigma_B \cup \Sigma_0 \cup \Sigma_\infty$. Here, $\Sigma_B$ stands for the mean wetted hull of a rigid body (ship or structure) or, more generally, a control surface that encloses a rigid body; $\Sigma_0$ is the portion of the mean free-surface plane located outside the "body" surface $\Sigma_B$; and $\Sigma_\infty$ joins $\Sigma_0$ and $\Sigma_D$ in the farfield. The unit vector $\mathbf{n} = (n^x, n^y, n^z)$ is normal to the boundary surface $\Sigma$ and points into the flow domain, as already noted. Thus, $\mathbf{n} = (0\,,0\,,-1)$ at the free surface $\Sigma_0$. The Green function $G$ in (2a) is presumed to vanish sufficiently rapidly in the farfield to nullify the contribution of the farfield boundary surface $\Sigma_\infty$. Thus, the contribution of $\Sigma_\infty$ is ignored, and the free surface $\Sigma_0$ is unbounded. Green's potential representation (2a) then becomes

$$\widetilde{C}\,\widetilde{\phi} = \int_{\Sigma_B} d\mathcal{A}\,(G\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla G) - \int_{\Sigma_0} dx\,dy\,(G\,\phi_z - G_z\,\phi) \tag{5}$$

Here, the $z$ axis is vertical and points upward, and the mean free surface is taken as the plane $z=0$. It is useful to use a Green function of the form

$$4\pi G = -1/r \pm 1/r_* + H \quad \text{with} \quad r_* = \sqrt{(x-\widetilde{x})^2 + (y-\widetilde{y})^2 + (z+\widetilde{z})^2} \tag{6}$$

$H(\mathbf{x}\,;\widetilde{\mathbf{x}})$ is harmonic within the flow region $\mathcal{D}$ (or a larger region that includes $\mathcal{D}$). Expressions (2b) and (6) show that $\widetilde{C} = 1$ if the flow-field point $\widetilde{\mathbf{x}}$ is located at the free surface $\Sigma_0$. Thus, (2b) becomes

$$\widetilde{C} = \left\{ \begin{matrix} 1 \\ 1/2 \\ 0 \end{matrix} \right\} \text{ if } \widetilde{\mathbf{x}} \text{ lies } \left\{ \begin{matrix} \text{in } \mathcal{D}\cup\Sigma_0 \\ \text{on } \Sigma_B \\ \text{outside } \mathcal{D}\cup\Sigma_B\cup\Sigma_0 \end{matrix} \right\} \tag{7}$$

**3. Local-flow and wave decomposition**

The harmonic function $H$ in the basic Green-function representation (6) can be decomposed into a local-flow component and a wave component. Thus, a Green function of the form

$$4\pi G = -1/r + L + W = R + W \tag{8}$$

is now considered. Here, the local-flow component $L$ and the wave component $W$ are presumed to satisfy the Laplace equations $\nabla^2 L = 0$ and $\nabla^2 W = 0$. The decomposition (8) is not unique. For instance, for diffraction-radiation of regular waves by an offshore structure, a particularly simple choice of Green function is defined in [ 1 ] as

$$R = -1/r - 1/r_* + 2/r_f \qquad W = -i\,f^2 e^{f^2 Z_*} \int_{-\pi}^{\pi} dt\,(1-\Theta)\,e^{-i\,\Phi} \tag{9a}$$

$$\text{where} \quad \left. \begin{matrix} r = \sqrt{h^2+Z^2} & r_* = \sqrt{h^2+Z_*^2} & r_f = \sqrt{h^2+Z_f^2} \\ h = \sqrt{X^2+Y^2} & X = \widetilde{x} - x & Y = \widetilde{y} - y \\ Z = \widetilde{z} - z & Z_* = \widetilde{z} + z & Z_f = Z_* - \sigma^R/f^2 \end{matrix} \right\} \tag{9b}$$

$$\Phi = f^2(X\cos t + Y\sin t) \qquad \Theta = \frac{\sinh(\Phi/\sigma^W) + i\,\sin(V/\sigma^W)}{\cosh(\Phi/\sigma^W) + \cos(V/\sigma^W)} \quad \text{with } V = f^2 Z_* \tag{9c}$$

and $-V/\sigma^W < C^W < \pi$. This Green function, which only involves elementary functions of real arguments, satisfies the free-surface condition $G_z - f^2 G = 0$ at $z = 0$ accurately in the farfield, but only approximately (to leading order) in the nearfield. The Rankine component $R$ involves three elementary free-space Rankine sources: a unit source at the singular point $\mathbf{x} = (x\,,y\,,z)$, a unit source at the mirror image $(x\,,y\,,-z)$ of $\mathbf{x}$ with respect to the mean free-surface plane $z = 0$, and a “double” sink (strength 2) at the point $(x\,,y\,,-z + \sigma^R/f^2)$. This arrangement of elementary point sources and sinks satisfies the linear free-surface boundary condition $G_z - f^2 G = 0$ at $z = 0$ to leading order, in both the nearfield and the farfield. The wave component is given by a one-dimensional Fourier superposition of elementary waves $e^{f^2 Z_* - i\,\Phi}$. The radiation condition is satisfied via the function $\Theta$; see [ 1 , 2 ]. More complicated Green functions that satisfy the linear free-surface boundary condition everywhere (in the nearfield as well as the farfield) can be used. These alternative free-surface Green functions are also of the form (8); see e.g. [ 3 , 4 ].

Using the Rankine-wave decomposition (8) of the Greeen function in (5), one obtains the representation $4\pi\widetilde{C}\,\widetilde{\phi} = \widetilde{\phi}^R + \widetilde{\phi}^W$ of the potential $\widetilde{\phi}$ at a flow-field point $\widetilde{\mathbf{x}}$. Here, the Rankine potential $\widetilde{\phi}^R$ and the wave potential $\widetilde{\phi}^W$ correspond to the Rankine and wave components $R$ and $W$ in (8), and $\widetilde{C}$ is given by (7). Thus, for diffraction-radiation of regular waves by an offshore structure, the potentials $\widetilde{\phi}^R$ and $\widetilde{\phi}^W$ are given by

$$\widetilde{\phi}^R = \int_{\Sigma_B} d\mathcal{A}\,(R\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla R) + \int_{\Sigma_0} dx\,dy\,[(R_z - f^2 R)\,\phi - R(\phi_z - f^2\phi)] \tag{10a}$$

$$\widetilde{\phi}^W = \int_{\Sigma_B} d\mathcal{A}\,(W\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla W) + \int_{\Sigma_0} dx\,dy\,[(W_z - f^2 W)\,\phi - W(\phi_z - f^2\phi)] \tag{10b}$$

Use of expressions (9) for the Rankine and wave components $R$ and $W$ in (10) yields expressions for the potentials $\widetilde{\phi}^R$ and $\widetilde{\phi}^W$ that only involve elementary functions of real arguments. These simple expressions for $\widetilde{\phi}^R$ and $\widetilde{\phi}^W$ are given in [ 5 ]. Integration over the free surface $\Sigma_0$ in (10) only needs to be performed over a finite nearfield portion of the unbounded free surface. In particular, the terms $R_z - f^2 R$ and $R$ in (10a) are $O(1/\rho^3)$ as $\rho \equiv \sqrt{x^2+y^2} \to \infty$.

**4. A seemingly paradoxical property**

The wave component $W$ in (8) satisfies the Laplace equation $\nabla^2 W = 0$, as already noted. It follows from (4) that the wave potential $\widetilde{\phi}^W$ in (10) is null, i.e. one has $\widetilde{\phi}^W \equiv 0$. However, the wave potential $\widetilde{\phi}^W$ is known to become exact in the horizontal farfield; specifically, one has $\widetilde{\phi}^W \sim 4\pi\widetilde{\phi}$ as $\rho \equiv \sqrt{\widetilde{x}^2+\widetilde{y}^2} \to \infty$. This seemingly contradictory result can be explained if the unbounded free surface $\Sigma_0$ is divided into a finite nearfield portion $\Sigma_0^{near}$ and an unbounded farfield portion $\Sigma_0^{far}$. Thus, the unbounded free surface is expressed as $\Sigma_0 = \Sigma_0^{near} \cup \Sigma_0^{far}$. Specifically, $\Sigma_0^{far}$ is taken here as the region $\rho_\infty < \rho$. Let $\widetilde{\phi}^R_{near}$ and $\widetilde{\phi}^W_{near}$ stand for the contributions of the finite nearfield boundary surface $\Sigma^{near} = \Sigma_B \cup \Sigma_0^{near}$ to the Rankine and wave potentials $\widetilde{\phi}^R$ and $\widetilde{\phi}^W$. One has

$$\left.\begin{aligned} &\widetilde{\phi}^W_{near} \approx 0 \quad \text{and} \quad \widetilde{\phi}^N_{near} \approx 4\pi\widetilde{\phi} \quad \text{for} \quad \rho \le \rho_{inner} < \rho_\infty \\ &\widetilde{\phi}^N_{near} \approx 0 \quad \text{and} \quad \widetilde{\phi}^W_{near} \approx 4\pi\widetilde{\phi} \quad \text{for} \quad \rho_\infty < \rho_{outer} \le \rho \end{aligned}\right\} \tag{11}$$

This property is numericallly illustrated and verified here by considering a simple axisymmetric flow generated by a pulsating point source. Specifically, consider the flow defined by the potential

$$\phi(\mathbf{x}) = G(\mathbf{a}\,;\mathbf{x}) \quad \text{with} \quad \mathbf{a} = (0\,,0\,,-0.1) \quad \text{and} \quad f = 2 \tag{12}$$

Here, $G$ stands for the free-surface Green function defined by (8) and (9) with $\sigma^R = 1$, $\sigma^W = 3$, $C^W = 2.3$. The flow due to the pulsating source (12) is considered for the unbounded flow region that is outside a spherical boundary surface $\Sigma_B$, taken as the half sphere $\sqrt{x^2+y^2+z^2} = 1$ with $z \le 0$. The flow $\widetilde{\phi}_{near}$ associated with (12) is defined by (10) in terms of the flux $\mathbf{n}\cdot\nabla\phi$ at $\Sigma_B$, the potential $\phi$ at $\Sigma_B \cup \Sigma_0^{near}$ and the pressure $\phi_z - f^2\phi$ at $\Sigma_0^{near}$. These three forcing terms, easily evaluated from (9), are depicted in Fig.1 along the line defined by $y = 0$ and

$$\left.\begin{aligned} &x = \sin(\pi t/2) \quad z = -\cos(\pi t/2) \quad \text{with } 0 \le t \le 1 \\ &x = t \quad z = 0 \quad \text{with } 1 \le t \le 18 \end{aligned}\right\} \tag{13}$$

Fig.1 shows that the pressure $\phi_z - f^2\phi$ at the free surface $\Sigma_0$ decays rapidly as $t$ increases, i.e. away from the (spherical) boundary surface $\Sigma_B$. The "input" potential $\widetilde{\phi}$ given by (12) and the Rankine potential $\widetilde{\phi}^R/(4\pi)$ and wave potential $\widetilde{\phi}^W/(4\pi)$, reconstructed using the boundary-integral representation (10), are depicted in Fig.2 and Fig.3 along the line

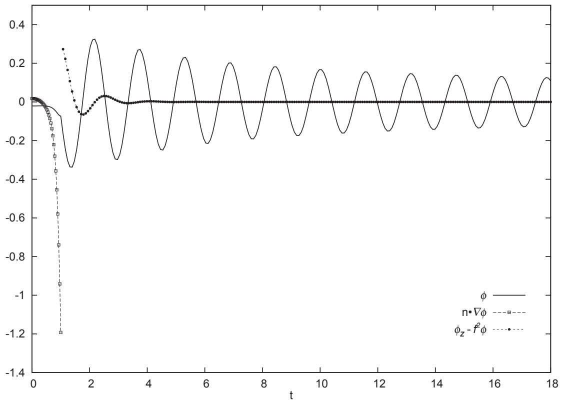
$$\left.\begin{aligned} &x = (1+\mu)\sin(\pi t/2) \quad z = -(1+\mu)\cos(\pi t/2) \quad \text{with } 0 \le t \le t^* \\ &x = t \quad z = -\mu \quad \text{with } t^* \le t \le 18 \end{aligned}\right\} \tag{14}$$

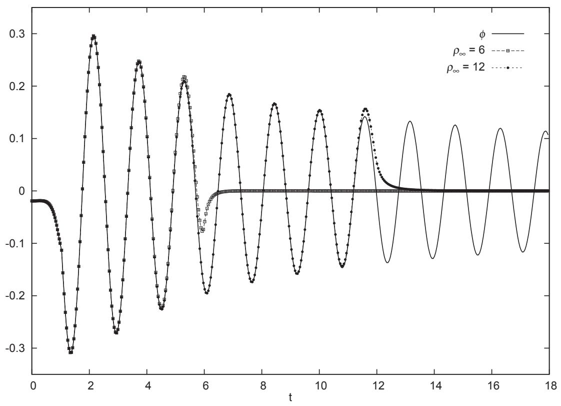
and $t^* = (2/\pi)\cos^{-1}[\,\mu/(1+\mu)\,]$. The line (14) is located inside the flow region, at a distance $\mu$ (taken equal to 0.03 here) from the line (13).

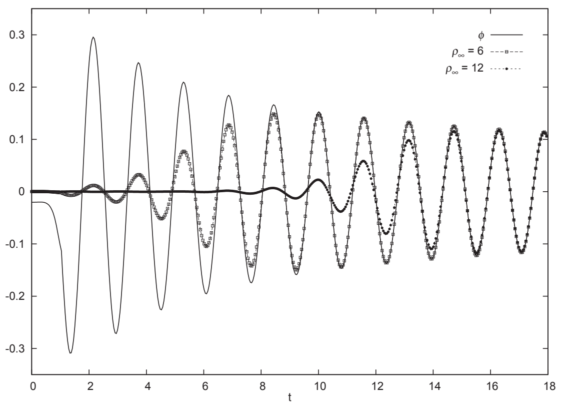
Fig.2 depicts the potential (12) and the Rankine potential $\widetilde{\phi}^R/(4\pi)$ with $\rho_\infty$ taken equal to 6 or 12. Thus, the radius $\rho_\infty$ of the nearfield region $\Sigma_0^{near}$ of the unbounded free surface is taken equal to 6 or 12 in Fig.2. This figure shows that $\widetilde{\phi}^R_{near} \approx 4\pi\widetilde{\phi}$ in an inner region $\rho \le \rho_{inner} < \rho_\infty$ and that $\widetilde{\phi}^R$ vanishes rapidly for $\rho_\infty < \rho$, in accordance with (11). The radius $\rho_{inner}$ of the inner region, roughly equal to 5 for $\rho_\infty = 6$ or to 11 for $\rho_\infty = 12$, increases as the radius $\rho_\infty$ of the nearfield region $\Sigma_0^{near}$ of the free surface increases, and one may presume that $\rho_{inner} \to \infty$ as $\rho_\infty \to \infty$. Similarly, Fig.3 depicts the potential (12) and the wave potential $\widetilde{\phi}^W/(4\pi)$ with $\rho_\infty$ taken equal to 6 or 12, as in Fig.1. Fig.2 shows that $\widetilde{\phi}^W_{near} \approx 4\pi\widetilde{\phi}$ in an outer region $\rho_\infty < \rho_{outer} \le \rho$ and that $\widetilde{\phi}^W_{near} \approx 0$ in an inner region $\rho \le \rho_{inner} < \rho_\infty$, in agreement with (11). Fig.3 shows that one has $\rho_{inner} \approx 1$ and $\rho_{outer} \approx 10$ for $\rho_\infty = 6$; and $\rho_{inner} \approx 6$ and $\rho_{outer} \approx 16$ for $\rho_\infty = 12$. Thus, the radius $\rho_{inner}$ of the inner region increases with the size of the nearfield free-surface region $\Sigma_0^{near}$, and one may presume that $\rho_{inner} \to \infty$ as $\rho_\infty \to \infty$.

## 5. Additional results

Implications of the property (11) and related numerical results given in Fig.2 and Fig.3 will be discussed at the Workshop. Alternative boundary integral representations suited for nearfield and farfield flows will also be presented, and numerically illustrated for wave diffraction-radiation by an offshore structure. A more detailed account of this study will be reported in [ 5 ] .




## References

[ 1 ] F. Noblesse, C. Yang (2004) “A simple Green function for diffraction-radiation of time-harmonic waves with forward speed”, *Ship Technology Research* **51**:35-52

[ 2 ] F. Noblesse, C. Yang (2006) “Elementary water waves”, *J. Engineering Mathematics*, in press

[ 3 ] F. Noblesse (1982) “The Green function in the theory of radiation and diffraction of regular water waves by a body”, *J. Engineering Mathematics* **16**:137-169

[ 4 ] B. Ponizy, F. Noblesse, M. Ba, M. Guilbaud (1994) “Numerical evaluation of free-surface Green functions”, *J. Ship Research* **38**:193-202

[ 5 ] C. Yang, R. Espinosa, R. Löhner, F. Noblesse (2007) “Alternative boundary-integral representations for wave diffraction radiation by offshore structures”, *Il J. Society Offshore and Polar Engineering*, submitted

## Figures

Fig.1 (top of right column) Flux $\mathbf{n}\cdot\nabla\phi$ at $\Sigma_B$ , potential $\phi$ at $\Sigma_B \cup \Sigma_0^{near}$ and pressure $\phi_z - f^2\phi$ at $\Sigma_0^{near}$. The free-surface pressure $\phi_z - f^2\phi$ vanishes rapidly in the farfield.

Fig.2 (center of right column) Input potential $\widetilde{\phi}$ given by (12) and Rankine potential $\widetilde{\phi}^R/(4\pi)$ for free-surface integration truncated at $\rho_\infty = 6$ and $\rho_\infty = 12$ . The Rankine potential $\widetilde{\phi}^R$ is exact in the nearfield and vanishes rapidly in the farfield.

Fig.3 (bottom of right column) Input potential $\widetilde{\phi}$ given by (12) and wave potential $\widetilde{\phi}^W/(4\pi)$ for free-surface integration truncated at $\rho_\infty = 6$ and $\rho_\infty = 12$ . The wave potential $\widetilde{\phi}^W$ is null in the nearfield and becomes exact in the farfield.