Wave trapping by freely-floating circular cylinders

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Abstract

Under the assumptions of the linearised theory of small-amplitude water waves, it is proved that a semi-immersed two-dimensional circular cylinder of radius a, floating on the surface of a fluid of infinite depth, and free to respond in heave and sway, is capable of totally reflecting an incident plane wave at a particular angular frequency. Numerically this is shown to occur at a single non-dimensional frequency given by $Ka \approx 1.126$ where $K = \omega^2/g$ (g is gravitational acceleration, $\omega/2\pi$ is frequency). This remarkable result is used to construct examples of motion trapped modes, involving pairs of identical circular cylinders each moving freely in heave and sway. Symmetric oscillations correspond to a motion trapped mode for a single cylinder next to a vertical wall; antisymmetric oscillations can be interpreted as motion trapped modes for a single catamaran structure with twin semi-circular hull profiles.

1. Introduction

As part of our abstract at last year's Workshop we showed how it was possible to construct motion trapped modes directly for a pair of heaving rectangular cylinders in two dimensions. Such modes are known to exist in the vicinity of a freely floating body and do not radiate their energy away. If such a body were to be displaced from rest and released, the ultimate motion would consist of a simple harmonic oscillation at the trapped mode frequency which persisted indefinitely. McIver & McIver (2006) were the first to construct pairs of identical bodies which exhibit motion trapping, using an inverse method. Our method was more direct. We first showed that a rectangular cylinder of certain dimensions, constrained to move in heave only, was capable of totally reflecting an incident wave at a particular frequency. Then wide-spacing arguments were used to show that an identical mirror-image of the cylinder could be placed at an appropriate distance from the first cylinder so as to trap the wave motion between them thereby constructing a motion trapped mode for a catamaran pair oscillating in heave. This was confirmed using full linear theory to construct motion trapped modes for two identical rectangular cylinders in free heave which need not be a large distance apart. It was also noted that for a freely-floating half-immersed *circular* cylinder constrained to move in heave only, the phenomenon of total reflection of an incident wave did not occur at any frequency. This was assumed to be because, unlike the rectangular cylinder, only one geometric parameter was available to be varied. We also commented in passing that even if the circular cylinder is allowed to respond in sway as well as heave to the incident wave, total reflection still did not occur. That statement was wrong. The following analysis shows that such a cylinder, free to respond in both heave and sway to an incident wave, (because of symmetry roll does not arise) does indeed reflect all the energy at a well-defined unique dimensionless wavenumber $Ka \approx 1.126$ where $K = \omega^2/g$ and a is the radius of the cylinder.

2. Scattering by a single freely-floating cylinder

Two-dimensional coordinates (x, y) are used with y vertically upwards and $y = 0$ aligned with the undisturbed free surface of a fluid of infinite depth. We assume a plane wave of frequency $\omega/2\pi$ is incident from $x = +\infty$ on the cylinder which responds with the same frequency. Then we may write $\Phi = \sum_{i=1}^{2} U_i \Phi_{R_i} + \Phi_S$ where Φ_S is the scattered potential due to a unit amplitude incident wave on the cylinder assumed to be held fixed, Φ_{R_i} and U_i are radiation potentials and component velocities, with $i = 1, 2$ corresponding to heave and sway respectively. We have

$$\Phi_{R_i} \sim \{\mathrm{sgn}(x)\}^{i-1} A_i \,\mathrm{e}^{\mathrm{i}K|x|-Ky}, \quad |x| \to \infty \tag{1}$$

$(i = 1, 2)$, so that A_i are the far-field radiated wave amplitudes, whilst $K = \omega^2/g$ is the wavenumber with g gravitational acceleration,

$$\Phi_S \sim \begin{cases} (g/\omega)(\mathrm{e}^{-\mathrm{i}Kx} + R\mathrm{e}^{\mathrm{i}Kx})\mathrm{e}^{-Ky}, & x \to \infty \\ (g/\omega)T\mathrm{e}^{-\mathrm{i}Kx-Ky}, & x \to -\infty \end{cases} \tag{2}$$

where R and T are the reflection and transmission coefficients for the fixed cylinder, dependent on frequency. It follows that

$$\Phi \sim \begin{cases} (g/\omega)(\mathrm{e}^{-\mathrm{i}Kx} + R_1\mathrm{e}^{\mathrm{i}Kx})\mathrm{e}^{-Ky}, & x \to \infty \\ (g/\omega)T_1\mathrm{e}^{-\mathrm{i}Kx-Ky}, & x \to -\infty \end{cases} \tag{3}$$

and combining (1)–(3) gives

$$\left.\begin{aligned} R_1 &= R + (\omega/g)\sum_{i=1}^{2} U_i A_i \\ T_1 &= T + (\omega/g)\sum_{i=1}^{2} (-1)^{i-1} U_i A_i \end{aligned}\right\}$$

so that

$$\left.\begin{aligned} R_1 + T_1 &= R + T + 2(\omega/g)U_1A_1 \\ R_1 - T_1 &= R - T + 2(\omega/g)U_2A_2. \end{aligned}\right\} \tag{4}$$

The equations of motion applied to the cylinder of mass M in each component are

$$\begin{aligned} -\mathrm{i}\omega M U_i &= F_{R_i} + F_{S_i} + F_i^{ext} \\ &= (\mathrm{i}\omega a_{ii} - b_{ii})U_i + F_{S_i} - \mathrm{i}\delta_{i1}\lambda U_i/\omega, \end{aligned} \tag{5}$$

where F_{S_i} is the vertical $(i = 1)$ and horizontal $(i = 2)$ exciting force on the fixed cylinder, and the last term is the hydrostatic restoring force in heave with $\lambda = 2a\rho g$. F_{R_i} are the forces on the cylinder due to the forced motion of the cylinder in heave $(i = 1)$ and sway $(i = 2)$, each decomposed into its added-mass and (non-negative) radiation damping components a_{ii} and b_{ii} respectively. Equation (5) may be written

$$b_{ii}(1 - \mathrm{i}C_i)U_i = F_{S_i}, \qquad (i = 1, 2) \tag{6}$$

where

$$C_i = \{(M + a_{ii})\omega^2 - \delta_{i1}\lambda\}/b_{ii}\omega. \tag{7}$$

$(i = 1, 2)$. If we now use the Haskind relation, the Newman relations, and the relation between the radiation damping coefficients and the far field amplitudes, we find that

$$(\omega/g)U_iA_i = -(R - (-1)^i T)/(1 - \mathrm{i}C_i), \tag{8}$$

$(i = 1, 2)$. Thus, we have

$$\begin{aligned} 2R_1 &= (R + T)(C_1 - \mathrm{i})/(C_1 + \mathrm{i}) \\ &\quad + (R - T)(C_2 - \mathrm{i})/(C_2 + \mathrm{i}) \\ 2T_1 &= (R + T)(C_1 - \mathrm{i})/(C_1 + \mathrm{i}) \\ &\quad - (R - T)(C_2 - \mathrm{i})/(C_2 + \mathrm{i}). \end{aligned}$$

It follows that $T_1 = 0$ provided

$$C_1C_2 + 1 - (C_1 - C_2)\chi = 0, \tag{9}$$

where $R/T = \mathrm{i}\chi$ and χ is real from the results $|R \pm T| = 1$ which arise from symmetry. It follows that $\chi = \pm|R|/|T|$. It is not obvious which sign to take but computations make it clear that for the cylinder, $\chi = -|R|/|T|$.

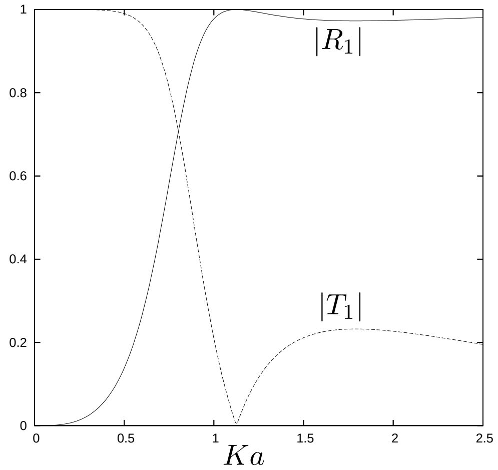

Figure 1: Reflected and transmitted wave amplitudes against Ka for a freely-floating cylinder

Since $C_2 > 0$, it is convenient to replace this by

$$f(Ka) \equiv (C_1 + \chi) + C_2^{-1}(1 - C_1\chi) = 0. \quad (10)$$

It is possible to prove that (10) does indeed have a real solution and hence that there exists a frequency at which $T_1 = 0$. A variety of asymptotic results are used in the proof of the result. Together these show that $f(Ka) \sim -1/(2Ka)$, $Ka \to 0$, and $f(Ka) \sim \frac{1}{4}\pi(Ka)^6(\pi^2 + 4)$ as $Ka \to \infty$.

Thus $f(Ka)$ changes sign and there must exist a value of $K = K_0$, say, for which $f(K_0a) = 0$. This remarkable result is confirmed by numerical calculations which show that there is just one value of $Ka = K_0a = 1.12594$ (to 5 d.p.'s) satisfying equation (10).

Fig. 1 shows the reflected and transmitted amplitudes, $|R_1|$ and $|T_1|$, for a freely-floating semi-immersed cylinder where it can be seen that $T_1 = 0$ at $K = K_0$.

We have shown, in contrast to the case of a half-immersed circular cylinder either fixed or free to move in heave *or* sway *only*, which always transmits some of the energy in an incident plane wave, that if such a cylinder is free in *both* heave *and* sway, there exists a unique frequency for which total reflection of the incident wave takes place.

On the basis of this result we can be confident that two such identical cylinders each free to move in heave and sway, and sufficiently far apart, can support a localised motion trapped mode between them by appealing to a wide-spacing argument. Thus, for two cylinders whose centres are separated by a distance $2b$, the wide spacing argument provides the approximation

$$Kb = -\tfrac{1}{2}\arg\{R_1\} + n\pi, \qquad n \in \mathbb{Z} \quad (11)$$

for an oscillation symmetric about a line bisecting the cylinders, at the frequency at which $|R_1| = 1$. An extra $\frac{1}{2}\pi$ is added to the right-hand side for antisymmetric modes.

3. Motion trapped modes for pairs of cylinders

As the above is only approximate, we need to consider the full unapproximated linear equations for a pair of freely floating circular cylinders, and determine the conditions under which a motion trapped mode can exist when they are allowed to move in heave under hydrostatic forces but in addition are free to move in sway. We approach this problem by considering the equivalent problem of a single cylinder next to a vertical 'wall' on which a Neumann/Dirichlet condition is placed for symmetric/antisymmetric oscillations. Ursell (1964) has solved the problem of the initial displacement of a single cylinder in the absence of the wall and our formulation follows his closely but with the inclusion of the wall. Thus we give the cylinder a small displacement $x_1(0)$ vertically before releasing it. In contrast to the Ursell problem, the wall will induce a horizontal motion which we have to allow for. We use Fourier transforms to solve the problem, and we find that the transformed equations of motion are

$$(M\omega^2 - \lambda\delta_{i1})U_i^w(\omega) = \lambda\delta_{i1}x_1(0) + \mathrm{i}\omega F_{R_i}^w(\omega), \quad (12)$$

$(i = 1, 2)$ where the superscript w refers to the presence of the wall. In terms of added mass and damping coefficients,

$$F_{R_i}^w(\omega) = (\mathrm{i}\omega a_{ii}^w - b_{ii}^w)U_i^w(\omega) + (\mathrm{i}\omega a_{ij}^w - b_{ij}^w)U_j^w(\omega),$$

$(i, j = 1, 2,\ i \neq j)$ in which $a_{12}^w = a_{21}^w$, $b_{12}^w = b_{21}^w$. Substitution into (12) gives

$$(C_i^w + \mathrm{i}) b_{ii}^w U_i^w(\omega) + (C^w + \mathrm{i}) b_{ij}^w U_j^w(\omega) = \delta_{1i} \lambda x_i(0)/\omega, \quad (13)$$

$(i, j = 1, 2,\ i \neq j)$ where

$$C_i^w = \{(M + a_{ii}^w)\omega^2 - \delta_{i1}\lambda\}/b_{ii}^w \omega, \quad (14)$$

$(i = 1, 2)$ and

$$C^w = a_{12}^w \omega / b_{12}^w = a_{21}^w \omega / b_{21}^w. \quad (15)$$

The time varying velocities are now given by

$$u_i^w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U_i^w(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega, \quad (16)$$

where (13) determines that

$$\left.\begin{aligned} U_1^w(\omega) &= \lambda \omega^{-1} x_1(0)(C_2^w + \mathrm{i}) b_{22}^w/\Delta \\ U_2^w(\omega) &= -\lambda \omega^{-1} x_1(0)(C^w + \mathrm{i}) b_{21}^w/\Delta \end{aligned}\right\}$$

with

$$\Delta = (C_1^w + \mathrm{i})(C_2^w + \mathrm{i}) b_{11}^w b_{22}^w - (C^w + \mathrm{i})^2 b_{12}^w b_{21}^w \quad (17)$$

Thus, the condition for there to be a motion trapped mode at a frequency $\omega = \omega_0$ is $\Delta = 0$ since the resulting pole on the real axis in (16) gives rise, at large times, to a dominant contribution proportional to $\mathrm{e}^{-\mathrm{i}\omega_0 t}$.

From (17), the condition $\Delta = 0$ can be split into its real imaginary parts to give two real conditions to be satisfied for a motion trapped mode. However, the general result

$$b_{11}^w b_{22}^w = b_{12}^w b_{21}^w, \qquad (\text{and } b_{12}^w = b_{21}^w)$$

can be proved for cylinder of arbitrary cross section, and this reduces the complex condition $\Delta = 0$ into two much simpler real conditions, namely

$$C_2^w = C_1^w = C^w.$$

First, we consider oscillations which are symmetric about the line bisecting the cylinders. The wide-spacing approximation,

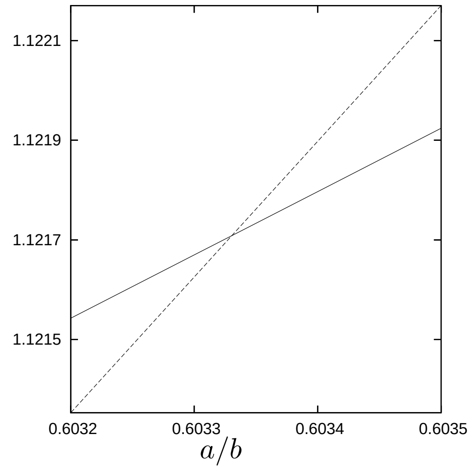

Figure 2: Values of Ka and a/b for the fundamental mode at the crossing point.

(11) predicts a motion trapped modes for $Ka = 1.12593$ and the sequence $a/b = 0.60484, 0.22505, \ldots$, $(n = 1, 2, \ldots)$. A careful computation of the exact method described above confirms that there is indeed motion trapped modes at values close to those predicted by the wide-spacing approximation. The exact parameters are detected by plotting curves in the $(Ka, a/b)$-plane along which the two real quantities $C_2^w - C^w$ and $C_1^w - C^w$ vanish. A motion trapped mode corresponds to the crossing of these two curves (as illustrated in fig. 2).

The fundamental mode (a pumping mode), furnishes exact values of $Ka = 1.12170$, $a/b = 0.60333$. The next $(n = 2)$ mode, in which a single wavelength approximately fits between the two cylinders, furnishes exact values of $Ka = 1.12590$ and $a/b = 0.22504$, extremely close to the wide-spacing values.

The first antisymmetric oscillation occurs at $Ka = 1.12612$ and $a/b = 0.32808$ (wide-spacing approximation gives $a/b = 0.32804$).

References

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2. Ursell, F. 1964 The decay of the free motion of a floating body. *J. Fluid Mech.* **19** 305–319.