On cancellation effects in the nonlinear evaluation of the orbital motion in ocean surface waves

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Novel methods and instrumentation are used to measure the elevation of the ocean surface over large spatial domain and time. There are many purposes of such experiments, where a few of them include: to document the wave population, study eventual formation of extreme waves, and enhance the understanding of the processes leading to wave breaking or white-capping that can be observed on the sea surface. An example is the airborne measurements taken in the Gulf of Tehuantepec experiment (GOTEX) off the Pacific coast of Mexico, see Melville et al. (2005). While the elevation of the sea is measured, the orbital velocity is not. It has been a desire to use the measured elevation as input to the calculation of the latter. The kinematics is precisely used as input for a complementary analysis of the conditions where white-capping and wave breaking are observed, where the latter is recorded by video-camera techniques. In the field measurements, the surface elevation is typically recorded over swaths that are 5 km long and 200 m wide, where the length direction is along the main propagation direction of the wave field. A typical resolution is 5 m in each direction which means that the surface elevation is recorded in 40000 nodes. A typical wavelength is 70 m which means an observational resolution of 14 nodes per wavelength. Given the measured wave elevation $\eta(\mathbf{x}, t)$, where $\mathbf{x} = (x_1, x_2)$ denotes horizontal coordinate and t time, one wants to calculate the fluid velocity of the wave field observed at the free surface. The observations also indicate the presence of a current, which should be accounted for in the analysis.

Analysis assuming potential flow. We assume that the wave field can be modelled by potential flow. Let ϕ denote the velocity potential. We seek a relation between the surface elevation, η, its time derivative, η_t, and the velocity potential, where the gradient of the latter gives the orbital velocity. In the case of a horizontal current $\mathbf{U}$, the kinematic boundary condition at the free surface gives

$$V = \phi_n \sqrt{1 + |\nabla\eta|^2} = \eta_t + \mathbf{U} \cdot \nabla\eta, \tag{1}$$

where n denotes the unit normal pointing out of the fluid and ∇ horizontal gradient. The solution of the Laplace equation connects ϕ_n to the wave potential at the free surface, denoted by $\widetilde{\phi}$, giving

$$\int_S \frac{1}{r}\frac{\partial\phi'}{\partial n'}\mathrm{d}S' = 2\pi\widetilde{\phi} + \int_S \widetilde{\phi}'\,\frac{\partial}{\partial n'}\left(\frac{1}{r}\right)\mathrm{d}S', \tag{2}$$

where S denotes the instantaneous free surface, r the distance between the evaluation point $(\mathbf{x}, y)$ and integration point $(\mathbf{x}', y')$, both at the free surface, i.e. $r =$

$\sqrt{R^2+(y-y')^2}$, where $R=|\mathbf{x}'-\mathbf{x}|$. y denotes the vertical coordinate. For convenience we write $\eta=\eta(\mathbf{x},t)$, $\eta'=\eta(\mathbf{x}',t)$, $\widetilde{\phi}=\widetilde{\phi}(\mathbf{x},t)$, $\widetilde{\phi}'=\widetilde{\phi}(\mathbf{x}',t)$, and so on. Using that $\phi_n \mathrm{d}S = V\mathrm{d}\mathbf{x}$ the integral equation becomes

$$\int_S \frac{V'}{r}\mathrm{d}\mathbf{x}' = 2\pi\widetilde{\phi} + \int_S \widetilde{\phi}'\sqrt{1+|\nabla\eta|^2}\frac{\partial}{\partial n'}\frac{1}{r}\mathrm{d}\mathbf{x}'. \tag{3}$$

Introducing the variable $D=(\eta'-\eta)/R$, corresponding to the difference in the wave elevation at two horizontal positions divided by the horizontal distance, we obtain that $1/r = 1/R + 1/R[(1+D^2)^{-\frac{1}{2}}-1]$.

Given the quantity $V=\eta_t+\mathbf{U}\cdot\nabla\eta$, the task is to evaluate the potential $\widetilde{\phi}$. Using Fourier transform, following Clamond and Grue (2001 §6), Grue (2002 §6), the integral equation may be brought on the form

$$\begin{aligned}
\widetilde{\phi} &= \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(\eta_t+\mathbf{U}\cdot\nabla\eta)}{k}\right\} + \eta\mathcal{F}^{-1}\{k\mathcal{F}(\widetilde{\phi})\} + \mathcal{F}^{-1}\left\{\frac{\mathrm{i}\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})\right\} \\
&+ \frac{1}{2\pi}\mathcal{F}\left\{\int \frac{\eta_t+\mathbf{U}\cdot\nabla\eta}{R}\left[(1+D^2)^{-\frac{1}{2}}-1\right]\mathrm{d}\mathbf{x}'\right\} \\
&+ \frac{1}{2\pi}\mathcal{F}\left\{\int \widetilde{\phi}'\left[(1+D^2)^{-\frac{3}{2}}-1\right]\nabla'\cdot\left[(\eta'-\eta)\nabla'\frac{1}{R}\right]\mathrm{d}\mathbf{x}'\right\},
\end{aligned} \tag{4}$$

where $\mathcal{F}$ denotes Fourier transform, $\mathcal{F}^{-1}$ inverse transform, $\mathbf{k}$ wavenumber in spectral space, and $k=|\mathbf{k}|$. The equation is solved for the wave potential by an iterative procedure, where a first approximation, $\widetilde{\phi}_1$, is obtained by computing the terms involving the given function $\eta_t+\mathbf{U}\cdot\nabla\eta$. A next approximation is obtained by computing the terms $\eta\mathcal{F}^{-1}\{k\mathcal{F}(\widetilde{\phi})\}+\mathcal{F}^{-1}\{\mathrm{i}(\mathbf{k}/k)\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})\}$ and $(1/2\pi)\mathcal{F}\{\int\widetilde{\phi}'\left[(1+D^2)^{-\frac{3}{2}}-1\right]\nabla'\cdot\left[(\eta'-\eta)\nabla'\frac{1}{R}\right]\mathrm{d}\mathbf{x}'\}$, in (4), using $\widetilde{\phi}_1$ as input, giving $\widetilde{\phi}_2$. The iteration is continued once more and has then converged. The horizontal orbital velocity is evaluated by $(u,v)=\nabla\widetilde{\phi}-(\eta_t+\mathbf{U}\cdot\nabla\eta+\nabla\eta\cdot\nabla\widetilde{\phi})/(1+|\nabla\eta|^2)\nabla\eta$.

Numerical sea. We illustrate the formulas by giving a numerical sea as input, and discuss results obtained from field data elsewhere. The surface elevation and its time derivative are obtained by superposition of several components of a spectrum, giving

$$\eta(\mathbf{x},t) = \sum_{m=-M/2,n=-N/2}^{M/2-1,N/2-1} A_{mn}\cos(\mathbf{k}_{mn}\cdot\mathbf{x}-\omega_{mn}t+\theta_{mn}), \tag{5}$$

where $A_{mn}=\sqrt{2S(\mathbf{k}_{mn})\Delta k_1\Delta k_2}$, $\mathbf{k}_{mn}=(\bar{k},0)+(m\Delta k_1, n\Delta k_2)$, $k_{mn}=|\mathbf{k}_{mn}|$, $\omega_{mn}=\sqrt{gk_{mn}}$, and θ_{mn} are random numbers on the interval $(0,2\pi)$. As spectrum we use the JONSWAP spectrum with $\gamma=5$. Directionality is obtained by the function $D(\alpha_{mn}) = (1/\beta)\cos^2(\pi\alpha_{mn}/2\beta)$, $\alpha_{mn}=\tan^{-1}(k_{2,n}/k_{1,m})$, with $\beta<0.7$. (5) is in the examples below evaluated at time $t=0$, with $H_s=10$ m and peak wavenumber $k_pH_s=0.467$ (corresponding to a wavelength of 135 m). A large wave event at $x_1=3900$ m, with $\eta_m=7.9$ m, $k_p\eta_m=0.367$, $\mathbf{U}=0$ and $\beta=0$, is illustrated in figure 1. The figure also plots η_t at $t=0$, with η_t estimated from

$$\eta_t(\mathbf{x},t) = \sum_{m=-M/2,n=-N/2}^{M/2-1,N/2-1} A_{mn}\omega_{mn}\sin(\mathbf{k}_{mn}\cdot\mathbf{x}-\omega_{mn}t+\theta_{mn}). \tag{6}$$

The linear estimate of the nonlinear η_t is used for illustrative purposes, see below. In this study the wave field is not integrated forward in time.

Cancellation effects. Calculations of the nonlinear orbital velocity exhibit the following features:

1) There is an almost perfect cancellation among the two quadratic terms in (4). Individually, the two terms are not small. This is particularly true in cases of large wave events, as illustrated in figure 2, upper panel. The results show that the two terms have same magnitude, but are of opposite sign. We note that the cancellation is perfect in the case of an elevation on the form $\eta(x_1, t) = A\cos(kx_1 - \omega t)$. The individual contributions to the orbital velocity from $[\eta\mathcal{F}^{-1}\{k\mathcal{F}(\widetilde{\phi})\}]_{x_1}$ and $[\mathcal{F}^{-1}\{\mathrm{i}(\mathbf{k}/k)\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})\}]_{x_1}$ are illustrated in figure 2, mid panel. The former term, which to leading order is approximated by $[\eta\eta_t]_{x_1}$ (with $\mathbf{U} = 0$), increases u_{max} from 5 m/s to 8 m/s at the wave crest, and reduces the strength of the velocity at the wave trough. The second term, $[\mathcal{F}^{-1}\{\mathrm{i}(\mathbf{k}/k)\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})\}]_{x_1}$, has the opposite effect, however. The (almost perfect) cancellation of the quadratic contribution in (4) means that the term $\int_S \widetilde{\phi}'(\partial/\partial n')(1/r)\mathrm{d}S'$ on the r.h.s. of (2) is always very small.

2) A coarse approximation to the potential is obtained by $\widetilde{\phi}_0 = \mathcal{F}^{-1}\{\mathcal{F}(\eta_t + \mathbf{U}\cdot\nabla\eta)/k\}$. The (nonlinear) orbital velocity calculated with this input, evaluated at the position of the free surface, is in fact very close to the nonlinear one, see figure 2, lower panel. The nonlinear velocity is in fact smaller than the semi-linear one in this example. This is explained by the multiplication of $1/r$ in the exact integral equation which is smaller than $1/R$ in the approximation.

3) The almost perfect cancellation among the quadratic terms provides a reason why nonlinear methods for wave computations of almost unidirectional seas converge rapidly (and are so successful).

References

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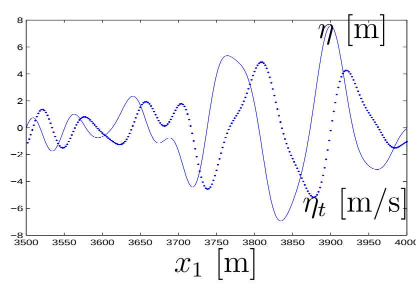

Figure 1: $-$ η [m], and dots η_t [m/s]. ($t = 0$).

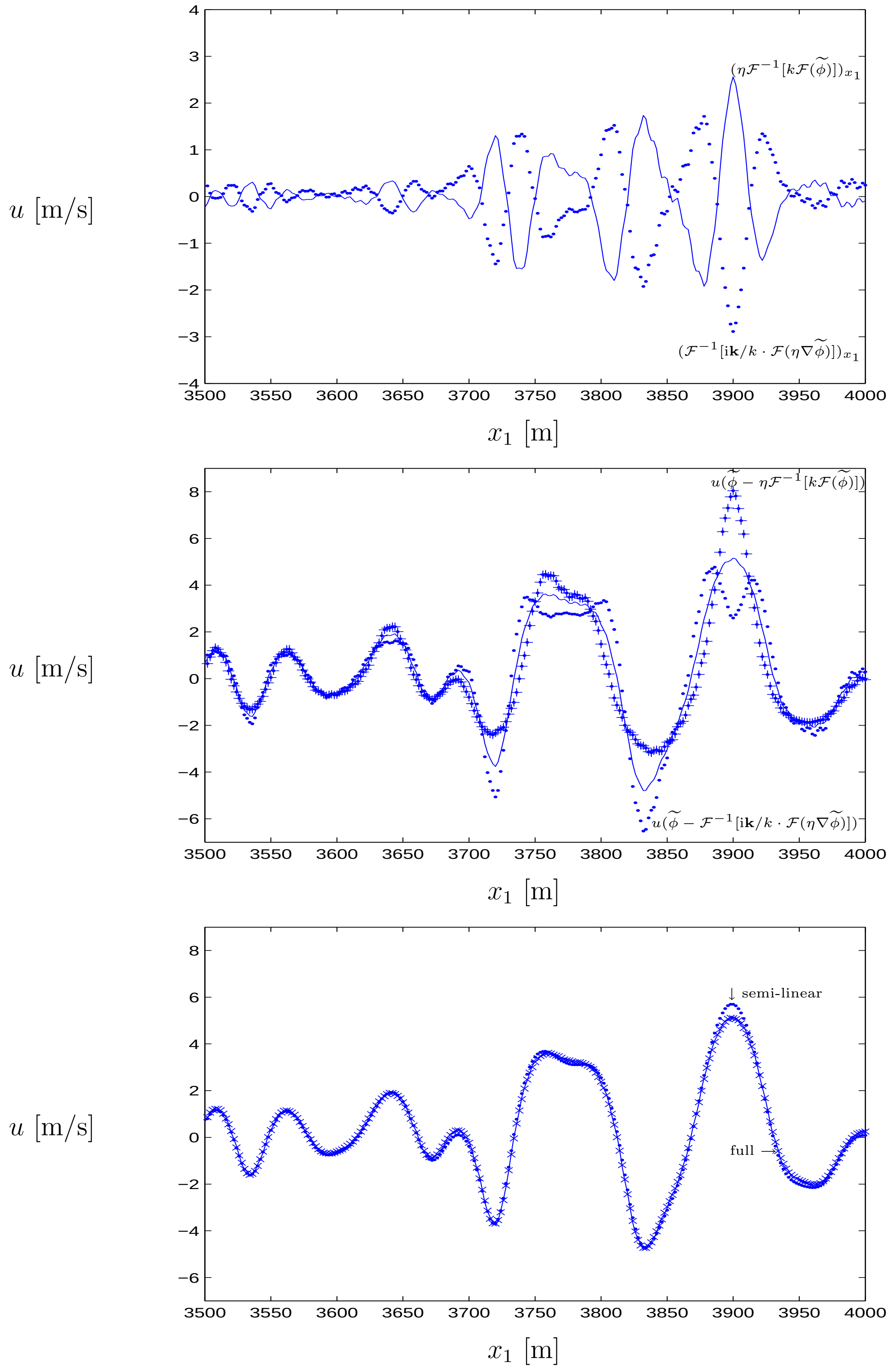

Figure 2:
Upper: – $(\eta\mathcal{F}^{-1}[k\mathcal{F}(\widetilde{\phi})])_{x_1}$; dots $(\mathcal{F}^{-1}[\mathrm{i}\mathbf{k}/k\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})])_{x_1}$ [m/s].
Mid: – exact u; dots: $u(\widetilde{\phi}-\eta\mathcal{F}^{-1}[k\mathcal{F}(\widetilde{\phi})])$; crosses: $u(\widetilde{\phi}-\mathcal{F}^{-1}[\mathrm{i}\mathbf{k}/k\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})])$ [m/s].
Lower: u [m/s] – full; x FFT-part; dots semi-linear.