TIME-DEPENDENT RESPONSE OF A HETEROGENEOUS ELASTIC PLATE FLOATING ON SHALLOW WATER

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1. Introduction

In global response analysis of large floating structures, the structure is often idealized as a homogeneous elastic plate. However, the actual floating structures (both the ice floes and manufactured platforms) show variability on all spatial scales. To the author's knowledge, the effect of structure heterogeneity of the floating elastic plate was investigated only for diffraction problem, by solving the linear hydroelastic problem for a single frequency (see, for example, the recent review [1]).

The aim of this paper is to consider the unsteady problem for a heterogeneous elastic plate floating on shallow water. Proposed methods may be used for any unsteady 2D problem of linear shallow-water theory, but here the scattering of localized surface wave by an elastic plate is considered. The solutions of this problem for a homogeneous plate were given in [2] for a flat bottom and in [3,4] for an uneven bottom. Action of an unsteady external load on a homogeneous elastic plate was considered for shallow water of uniform depth in [5].

2. Mathematical formulation

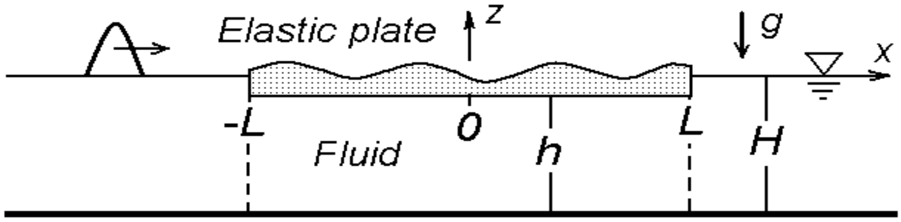

Figure shows a schematic diagram of the problem. An elastic plate floats on the surface of an inviscid incompressible fluid layer. The plate is infinite in the y-direction, so that only the x-and z-directions are considered. The x-direction is horizontal, the positive z-axis points vertically up, and the plate covers the region $-L \leq x \leq L$. The water is of uniform depth H which is small enough that the water may be approximated as shallow. The fluid motion is considered potential. The velocity potentials describing the fluid motion in the regions under the plate and outside the plate are denoted by $\phi_1(x,t)$ and $\phi_2(x,t)$, respectively, where t is time. Under the plate, fluid depth is equal to $h = H - d$, where d is the draft of the plate. For simplicity we assume that the draft is uniform along the plate.

A deflection of a heterogeneous elastic plate $w(x,t)$ is described by the equation:

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 w}{\partial x^2}\right) + m(x)\frac{\partial^2 w}{\partial t^2} + g\rho w + \rho\frac{\partial \phi_1}{\partial t} = 0 \quad (|x| \leq L), \tag{1}$$

where $D(x)$ is the flexural rigidity of the plate, $m(x)$ is the mass per unit length of the plate, ρ is the fluid density, and g is the gravity acceleration. According to Archimedes' principle, the draft of the plate is equal

$$d = \frac{1}{2\rho L}\int_{-L}^{L} m(x)\mathrm{d}x.$$

The functions $D(x)$ and $m(x)$ must have an integrable second derivatives, be piecewise continuous themselves, and must have a piecewise continuous first derivatives.

According to linear shallow-water theory, the following relation is valid:

$$\frac{\partial w}{\partial t} = -h\frac{\partial^2 \phi_1}{\partial x^2} \quad (|x| \leq L). \tag{2}$$

In the free-water regions, the velocity potential $\phi_2(x,t)$ satisfies the equation

$$\frac{\partial^2 \phi_2}{\partial t^2} = gH \frac{\partial^2 \phi_2}{\partial x^2} \quad (|x| > L). \tag{3}$$

The displacement of the free surface $\eta(x,t)$ is determined from the relation $\eta = -g^{-1}\partial\phi_2/\partial t$. If $|x| = L$, the matching conditions (continuity of pressure and mass) should be satisfied:

$$\frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_2}{\partial t}, \quad \frac{\partial \phi_1}{\partial x} = \frac{H}{h}\frac{\partial \phi_2}{\partial x}. \tag{4}$$

The edges of the plate are free of shear force and bending moment. It is assumed that $D'(x) = 0$ at $|x| = L$, where the prime denotes differentiation with respect to x. Then free-edge conditions have the form:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \quad (|x| = L).$$

It is assumed, that at the initial time the plate and fluid at $x > -L$ are at rest. At $x < -L$, the localized displacement of the free surface $\eta_0(x,t) = f(x - \sqrt{gH}t)$ travels to the right. The function $f(\xi)$ is different from zero only at $|\xi| < c$. At $t = 0$ this wave reaches the left edge of the plate and the plate begins to undergo a complex bending motion in response to the incoming wave. Consequently, the initial conditions have the form:

$$\eta = \eta_0(x), \quad \frac{\partial \phi_2}{\partial t} = -g\eta_0(x) \quad (x < -L,\ t = 0); \quad w = \eta = \frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_2}{\partial t} = 0 \quad (x > -L,\ t = 0). \tag{5}$$

Non-dimensional variables are used below: L is taken as the length scale and $\sqrt{L/g}$ as the time scale.

Two methods are presented for the solution of this problem. Both methods are based on the expansion of the plate motion in the dry modes of vibration of the free plate. The eigenfunctions of a heterogeneous plate are used in the first method (Sect. 3), whereas the eigenfunctions of a homogeneous plate are used in the second method (Sect. 4).

3. Mode expansions of a heterogeneous plate

The plate deflection is sought in the form

$$w(x,t) = \sum_{n=0}^{\infty} a_n(t)\Psi_n(x). \tag{6}$$

Here the functions $a_n(t)$ are to be determined and the functions $\Psi_n(x)$ are solutions of the spectral problem:

$$(D(x)\Psi_n'')'' = \mu_n^4 m(x)\Psi_n \quad (|x| \le 1), \quad \Psi_n'' = \Psi_n''' = 0 \quad (|x| = 1). \tag{7}$$

Because this is a self-adjoint problem, all the eigenvalues are nonnegative: $\mu_n \ge 0$ $(n = 0, 1, 2, ...)$. The eigenfunctions $\Psi_n(x)$ constitute a complete and orthogonal (in a generalized sense) system:

$$\int_{-1}^{1} m(x)\Psi_j(x)\Psi_n(x)\mathrm{d}x = \delta_{jn},$$

where δ_{jn} is the Kroneker symbol.

We substitute expansion (6) into (1) and initial conditions (5), multiply the obtained relations by $\Psi_k(x)$, and integrate them over x from -1 to 1. Using the properties of the functions $\Psi_n(x)$, we obtain the set of ordinary differential equations (ODE's)

$$\ddot{a}_k + \sum_{n=0}^{\infty}(\mu_k^4\delta_{kn} + K_{nk})a_n + F_k(t) = 0$$

with the initial conditions $a_n(0) = \dot{a}_n(0) = 0$, where

$$K_{nk} = \int_{-1}^{1} \Psi_n(x)\Psi_k(x)\mathrm{d}x, \quad F_k(t) = \int_{-1}^{1} \Psi_k \frac{\partial \phi_1}{\partial t}\mathrm{d}x,$$

and an overdot denotes differentiation with respect to time.

A solution for $\phi_1(x,t)$ is sought in the form

$$\phi_1(x,t) = -\frac{1}{h}\left[\sum_{n=0}^{\infty} \dot{a}_n(t)\Phi_n(x) + Q(x,t)\right],$$

where the functions $\Phi_n(x)$ satisfy the equation $\Phi_n''(x) = \Psi_n(x)$. The function $Q(x,t)$ is to be determined. According to the Eq. (2) and the initial conditions (5), the function $Q(x,t)$ has the form

$$Q(x,t) = xU(t) + V(t), \quad U(0) = V(0) = 0.$$

The functions $U(t)$ and $V(t)$ are determined from the matching conditions (4).

The solution for $\phi_2(x,t)$ at $x < -1$ is sought in the form $\phi_2(x,t) = \phi_0(x,t) + \psi(x,t)$, where $\phi_0(x,t)$ is the velocity potential of coming wave and is determined from the equation $\partial\phi_0/\partial x = \eta_0/\sqrt{H}$. The function $\psi(x,t)$ defines the velocity potential of the reflected surface waves. According to Eq. (3), the solution for $\psi(x,t)$ has the form

$$\psi(x,t) = \begin{cases} A((x+1)/\sqrt{H} + t), & -(1+\sqrt{H}t) < x < -1, \\ 0, & x < -(1+\sqrt{H}t), \end{cases}$$

where the function $A(\xi)$ is unknown and should be determined.

In a similar manner, we can seek the solution for $\phi_2(x,t)$ at $x > 1$, which defines the velocity potential of the transmitted surface waves

$$\phi_2(x,t) = \begin{cases} B(t - (x-1)/\sqrt{H}), & 1 < x < 1+\sqrt{H}t, \\ 0, & x > 1+\sqrt{H}t, \end{cases}$$

where the function $B(\xi)$ is to be determined.

Using the matching conditions (4), we have

$$\dot{A} = -\frac{1}{\sqrt{H}}\left[\sum_{n=0}^{\infty} R_n^- \dot{a}_n(t) + U(t)\right] - \alpha(t), \quad \dot{B} = \frac{1}{\sqrt{H}}\left[\sum_{n=0}^{\infty} R_n^+ \dot{a}_n(t) + U(t)\right],$$

where $R_n^{\pm} = \Phi_n'(\pm 1)$, $\alpha(t) = \eta_0(-1,t)$.

The final set of ODE's has the form

$$\sum_{n=0}^{\infty}\left\{\delta_{nk} - \frac{1}{h}\left[D_{nk} + \frac{1}{2}\left(P_n^-(L_k - M_k) - P_n^+(L_k + M_k)\right)\right]\right\}\ddot{a}_n + \frac{1}{2\sqrt{H}}\sum_{n=0}^{\infty}\left[R_n^+(L_k+M_k) + R_n^-(L_k - M_k)\right]\dot{a}_n +$$

$$+\sum_{n=0}^{\infty}\left(\mu_k^4\delta_{nk} + K_{nk}\right)a_n + \frac{U}{\sqrt{H}}L_k + \alpha(L_k - M_k) = 0,$$

$$\dot{U} = \frac{1}{2}\sum_{n=0}^{\infty}\ddot{a}_n(P_n^- - P_n^+) - h\left[\frac{1}{2\sqrt{H}}\sum_{n=0}^{\infty}(R_n^+ + R_n^-)\dot{a}_n + \frac{U}{\sqrt{H}} + \alpha\right],$$

where

$$D_{nk} = \int_{-1}^{1}\Phi_n(x)\Psi_k(x)\mathrm{d}x, \quad M_k = \int_{-1}^{1}\Psi_k(x)\mathrm{d}x, \quad L_k = \int_{-1}^{1}x\Psi_k(x)\mathrm{d}x, \quad P_n^{\pm} = \Phi_n(\pm 1).$$

Once the $a_n(t)$ and $U(t)$ are determined, we can find all characteristics of motion of the fluid and the elastic plate.

Unfortunately, free natural modes of vibration can only be determined analytically for very few heterogeneities, for example, if the flexural rigidity and mass are piece-wise constant functions. More general heterogeneities demand a numerical approach.

4. Mode expansions of a homogeneous plate

For the solution of the unsteady problem given in Sect. 2 we can use also the expansion of the plate deflection of the heterogeneous plate in the eigenfunctions of vibrations of a free-edges homogeneous plate

$$w(x,t) = \sum_{n=0}^{\infty} b_n(t) W_n(x), \tag{8}$$

where the functions $b_n(t)$ are to be determined and the functions $W_n(x)$ are solutions of the spectral problem:

$$W_n^{(IV)} = \lambda_n^4 W_n \quad (|x| \le 1), \quad W_n'' = W_n''' = 0 \quad (|x| = 1).$$

The solution of this problem is well known (see, for example, [3-5]).

Using the expansion (8), we obtain from Eq. (1) the set of ODE's by analogy with Sect. 3:

$$\sum_{n=0}^{\infty} \Big[S_{kn} \ddot{b}_n + (\delta_{kn} + T_{kn}) b_n \Big] + Z_k(t) = 0$$

with the initial conditions $b_n(0) = \dot{b}_n(0) = 0$, where

$$S_{kn} = \int_{-1}^{1} m(x) W_k(x) W_n(x) \mathrm{d}x, \quad T_{kn} = \int_{-1}^{1} D(x) W_k''(x) W_n''(x) \mathrm{d}x, \qquad Z_k(t) = \int_{-1}^{1} W_k \frac{\partial \phi_1}{\partial t} \mathrm{d}x.$$

A solution for $\phi_1(x,t)$ is sought in the same form as in the unsteady problem for the homogeneous plate [5]. This method is sufficiently versatile because it may be applied to plates with the arbitrary distributions of $D(x)$ and $m(x)$.

5. Discussion

A correlation between two proposed methods is demonstrated with piece-wise constant functions for the flexural rigidity and mass:

$$(D(x), m(x)) = \begin{cases} (D_1, m_1), & (|x| < x_1), \\ (D_2, m_2), & (x_1 < |x| < 1), \end{cases}$$

where $0 < x_1 < 1$. The solution of the diffraction problem for this floating plate was given in [6].

In this case, the solution of the spectral problem (7) is obtained in the analytical form. The plate deflections and wave motions of the fluid have been calculated for various values D_1, D_2, m_1, m_2 and x_1 and will be presented at the Workshop.

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