Shallow-water Wave Propagation over Slowly-varying Bathymetry using High-order Boussinesq Equation

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Extended Abstarct

Wave effects are to be considered seriously for the design of coastal structures such as breakwaters, harbor, and moored-floaters. The performance analysis of structures in shallow water generally requires a number of possible wave conditions in order to determine the rational design criteria. Wave conditions can be generated by suitable mathematical models or numerical methods which must cope with various wave deformations such as shoaling, refraction, diffraction and reflection of waves propagating from deep water to shallow water.

Boussinesq models are well known as the most accurate method for describing the propagation of non-linear shallow water waves near coastal regions. The major effectiveness of Boussinesq formulations is the incorporation of dynamic properties into horizontal dimensions by eliminating the vertical coordinate. It significantly reduces the computational burden relative to three-dimensional methods and thus makes wave simulations in a wide coastal region practically feasible. In the classical form, Boussinesq equation represents a shallow water approximation to the exact Laplace problem which incorporates the balance between lowest-order dispersion and lowest-level non-linearity. Many researchers have tried to derive modified forms of the classical Boussinesq equation over last decades and a number of enhanced higher-order Boussinesq equations have been derived improving the dispersion and non-linearity as well as flow kinematics and dynamics (e.g. Madsen and Sorensen, 1992; Nwogu, 1993; Wei et al., 1995, Agnon et al., 1999; Gobbi et al., 2000; Wu, 2001; Madsen et al., 2002,

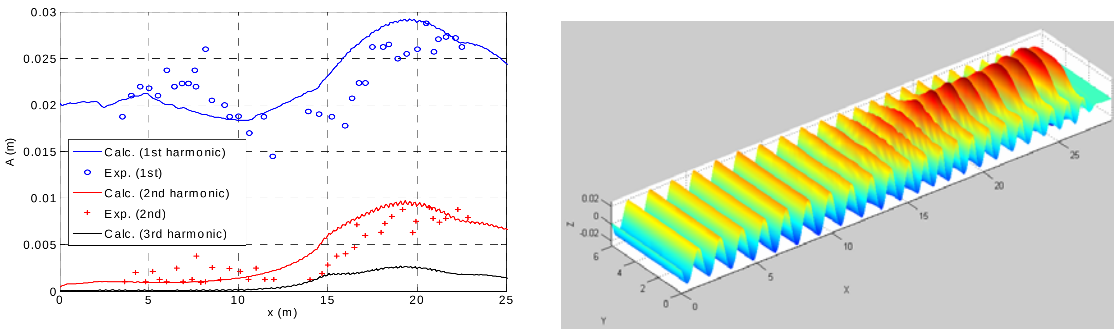
2003). Among these, the formulation of Madsen et al. (2002, 2003) is most capable of treating highly non-linear waves to $kh = 25$ for dispersion, with accurate velocity profiles up to $kh = 12$.

In this work, numerical simulations are carried out for waters of slowly-varying bathymetry based on Madsen's formulation, which represents basically a fully non-linear time-stepping method of the exact free surface boundary conditions with the kinematic bottom condition. Hereby some numerical schemes are introduced in order to describe shoaling, refraction and also irregular waves properly. The connection between the vertical velocity, $\tilde{w}$, and the horizontal velocity, $\tilde{\mathbf{u}}$, at the free-surface is established by expanding the velocity solution which satisfies the Laplace equation in the interior domain. The present method applies a truncated, Padé-enhanced Taylor series expansion of the velocity about an arbitrary level $z = \hat{z}$ in the fluid layer. The utility velocities $\hat{\mathbf{u}}^*$ and $\hat{w}^*(at\ z = \hat{z})$ are introduced for an approximate solution of the Laplace equation. A system of partial difference equations(PDE) is obtained by combining the expanded velocity and the bottom boundary condition. After having solved this system by the finite difference method, the utility velocities $\hat{\mathbf{u}}^*$ and $\hat{w}^*$ are obtained and then the velocities at the free surface are determined. Finally, the velocity and free surface elevation is updated by time-stepping the non-linear free-surface boundary condition.

Various numerical schemes are used for solving the PDE system by specifying the boundary conditions. As for the wave propagation problem, a wave-maker is necessary in order to generate the prescribed incident waves. A sponge layer is applied at the open boundaries to absorb the out-going waves. Both the wave maker and the wave absorber are set up with the concept of relaxation zone which is an extended region of the interior domain. A symmetric or reflection boundary condition is additionally applied. The reflection boundary is imposed by flipping FDM coefficients at the selected nodes versus the local boundary evenly (Neumann boundary) or oddly (Dirichlet boundary). Two kinds of matrix inversion method are considered. GMRES (Generalized Minimal Residual) may be the most interesting method for solving a large, non-symmetric matrix iteratively. Most calculations herein are performed by GMRES method. Recently, Bi-CG (Bi-Conjugate Gradient) has been proposed for solving non-symmetrical linear systems, and its excellent convergence behavior has been confirmed in many numerical computations. The required memory for GMRES is about $O(17N)$, while it is only $O(8N)$ for Bi-CG method, where N is the order of linear systems. On account of practical applications, Bi-CG method is used when large memory is needed. For the pre-conditioning, ILU factorization is applied in each sub-time-step in order to save memory

The linear wave shoaling is fundamentally imbedded in the governing equation for wave propagation over varying bathymetry. In order to verify the linear shoaling, the following test case is considered. At the seaward boundary, the water depth is $13m$. The bottom is flat for the first $10m$ from the boundary, while it has a constant slope of 1/50 from 10m to 600m distance. Finally from 600m to 650m, the bottom is flat again with a water depth of 1.2m. As a typical example for case of waves coming from deep water to shallow water, a wave with period of 8.0s is chosen which corresponds to h/L_0 varying from 0.13 to 0.002. The computed maximum elevations agree well with the shoaling curve obtained by the exact shoaling equation over the entire path. Based on this simulation result, it is concluded that the accuracy of our numerical model is acceptable in the aspect of linear wave shoaling.

Whalin's experiment (1971) is extensively cited in the literature to validate numerical wave models involving both the non-linear refraction and the shoaling. The bottom topography has a shoaling region acting as a concave lens. For waves of $\mathrm{T{=}1s}$, it corresponds to $\mathrm{kh{=}1.913}$ and $\mathrm{ka{=}0.0816}$. The simulation runs for 50 seconds to ensure the steady state. The relative amplitudes of 1st and 2nd harmonics are obtained along the centerline. The harmonic analysis is made from the time series of surface elevation from the last 20 seconds. Comparisons between the numerical results and experimental results are quite good as shown in the figure below. It is confirmed from the simulation that the present model has the capability of describing the non-linear shoaling reasonably.



Simulation Result for Nonlinear Wave Shoaling (Left-Computed and Measured Harmonic Amplitudes; Right-Snapshot of Surface Elevation)

Irregular waves in water of constant depth are simulated to study the generation and absorption characteristics of irregular waves in shallow water. It is to note that the wave component corresponding to the peak frequency of the simulated wave spectrum is classified to shallow water waves ($kh \leq 1.0$). The irregular waves are imposed at a boundary using JONSWAP spectrum which has the peak period of 12s and significant wave height of 3m. The simulation shows that the input and the generated spectrum match each other quite well with relative errors less than 1% in the sense of the total energy. Based on this result, it may be concluded that there is no significant numerical dissipation in the interior domain and the absorption layer works well for waves of all frequencies.

The present work can be summarized as follows: A high-order Boussinesq equation based on the Padé expansion is modeled to calculate the fully non-linear and highly dispersive waves. Computations are made by finite difference approximation of the Boussinesq model. Some numerical schemes are implemented to treat the boundary condition rationally, to smooth highly-oscillating numeral results properly and to inverse the sparse matrix effectively. As a result, it is possible to reduce the computational burden and it enables wave simulations in a wide coastal region. Several cases with different bottom topography are simulated. The present results show a good agreement with the experimental (or exact solution) counterpart for the linear and non-linear shoaling characteristics. Therefore the numerical method presented in this paper may be useful for predicting the propagation of shallow water waves in large domains with arbitrary bottom topology.