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Energy identity for flexural-gravity wave scattering due to a crack in a floating ice sheet in two-layer fluid in the context of blocking dynamics

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Highlights

- Blocking dynamics of flexural-gravity wave motion in two-layer fluid is revisited.
- The movement of the loci of the roots of the dispersion relation in the complex plane with an increase in wave frequency within primary and secondary blocking is demonstrated.
- Energy identity is established for scattering of flexural gravity waves due to a crack in a floating ice sheet in the frequency band where the dispersion relation possesses four positive real roots.

1. Introduction

A typical feature of the polar regions is the ice covered ocean surface in which waves generated in open sea could penetrate far into the ice-covered sea surface. The characteristics of the surface gravity waves will change due to its interaction with the ice sheet to generate the flexural gravity waves. One of such problems is the study on the scattering of flexural gravity waves due to a crack in an infinitely extended floating ice sheet [1, 2, 3]. Bhattacharjee and Sahoo [4] investigated the scattering of flexural gravity waves due to an infinitely extended floating ice sheet in a two-layer fluid having an interface in the absence of lateral compressive force. Das et al. [5, 6] demonstrated that in the presence of compressive force, flexural-gravity wave blocking occurs in which the group velocity vanishes in both the cases of single-layer and two-layer fluid domains respectively. In the frequency band between primary and secondary blocking in a two-layer fluid having an interface, flexural-gravity wave propagates with negative energy flux where the dispersion relation associated with flexural-gravity wave often possesses four positive real roots. Out of the four roots, three roots occur in the surface mode and one in the interface mode when blocking occurs in the surface mode, whilst one root occurs in the surface mode and three in the internal mode when blocking occurs in the interface mode. On the other hand, the dispersion relation possesses two real roots (one in surface mode and other one in the interface mode) outside the frequency band. In the proposed study, flexural-gravity wave scattering is considered in the presence of compressive force to further understand the loci of the roots of the dispersion relation in the complex plane and establish the energy identity when multiple propagating modes occurs within a frequency band during wave blocking.

1. Mathematical formulation

In the present study, we consider the flexural gravity wave scattering by a straight line crack in a floating ice sheet in the presence of lateral compressive force in a two-layer fluid of infinite depth. The physical problem is analyzed under the assumption of linear water wave theory and small amplitude structural response of the floating ice sheet. The ice-sheet is modelled as two semi-infinite elastic plates separated by a crack at $(x, y) = (0, 0)$ (see Figure 1). The physical problem is considered in the two-dimensional Cartesian coordinate system. The upper fluid layer of constant density ρ_1 occupies the region $-\infty < x < \infty$, $0 < y < h$, with $y = 0$ being the mean ice covered surface. The lower layer fluid density $\rho_2 (> \rho_1)$ occupies the

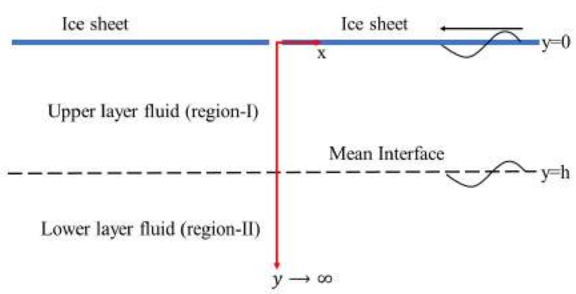


Figure 1: Floating ice sheet having a crack in a two-layer fluid

region $-\infty < x < \infty$, $h < y < \infty$ with the interface being at at $y = h$. The fluid is assumed to be inviscid and incompressible, and the flow is irrotational and simple harmonic in time with angular frequency ω which ensures the existence of a velocity potential $\Phi(x, y, t)$ of the form $\Phi(x, y, t) = \text{Real}\{\phi(x, y)e^{-i\omega t}\}$. Assuming that the x-axis is in the horizontal direction and the y-axis in the vertical downward positive direction. Thus, the spatial velocity potential $\phi(x, y)$ satisfies the partial differential equation

$$\nabla^2\phi = 0, \quad \text{in the fluid region.} \tag{1}$$

On the ice covered mean free surface, the boundary condition is given by

$$\left(D\frac{\partial^4}{\partial x^4} + Q\frac{\partial^2}{\partial x^2} + 1\right)\frac{\partial\phi(x, y)}{\partial y} + K\phi(x, y) = 0 \quad \text{on} \quad y = 0,\, x \in (-\infty, \infty) \setminus \{0\}, \tag{2}$$

where $D = EI/(\rho_1 g - d\rho_p\omega^2)$, $Q = N/(\rho_1 g - d\rho_p\omega^2)$, $K = \rho_1\omega^2/(\rho_1 g - d\rho_p\omega^2)$, $EI = Ed^3/12(1-\nu)$, E is Young's modulus, d is plate thickness, ρ_p is plate density, ν is Poisson's ratio, N (Newton$\cdot m^{-1}$) is the uniform compressive stress. The linearized condition at the mean interface is given by (as in [4])

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=h-} = \left.\frac{\partial\phi}{\partial y}\right|_{y=h+}, \quad s\left.\left(\frac{\partial\phi}{\partial y} + K\phi\right)\right|_{y=h-} = \left.\left(\frac{\partial\phi}{\partial y} + K\phi\right)\right|_{y=h+}, \quad -\infty < x < \infty, \tag{3}$$

where $s = \rho_1/\rho_2$. The condition at the bottom bed is given by

$$\phi, |\nabla\phi| \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty \tag{4}$$

Moreover, across the interface boundary between the two plate-covered regions, the continuity of velocity and pressure yields

$$\phi_x(0+, y) = \phi_x(0-, y) \text{ and } \phi(0+, y) = \phi(0-, y) \quad \text{for} \quad 0 < y < \infty. \tag{5}$$

Further, assuming the free-edge conditions (zero bending moment and shear stress) are complied near the crack, the velocity potential $\phi(x, y)$ satisfies the following edge conditions:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi}{\partial y}\right) = 0 \quad \text{and} \quad D\frac{\partial^3}{\partial x^3}\left(\frac{\partial\phi}{\partial y}\right) + Q\frac{\partial^2\phi}{\partial x\partial y} = 0 \quad \text{as} \quad (x, y) \longrightarrow (0\pm, 0). \tag{6}$$

Finally, the velocity potential satisfies the far-field radiation condition of the form

$$\phi(x, y) = \begin{cases} \sum_{n=I}^{II}\left(e^{-\mathrm{i}k_n x} + R_n e^{i\epsilon_n k_n x}\right)F_n(y) + \sum_{n=3}^{4} R_n e^{i\epsilon_n k_n x}F_n(y) & \text{as} \quad x \longrightarrow \infty, \\ \sum_{n=I}^{II} T_n e^{-\mathrm{i}\epsilon_n k_n x}F_n + \sum_{n=3}^{4} T_n e^{-\mathrm{i}\epsilon_n k_n x}F_n & \text{as} \quad x \longrightarrow -\infty, \end{cases} \tag{7}$$

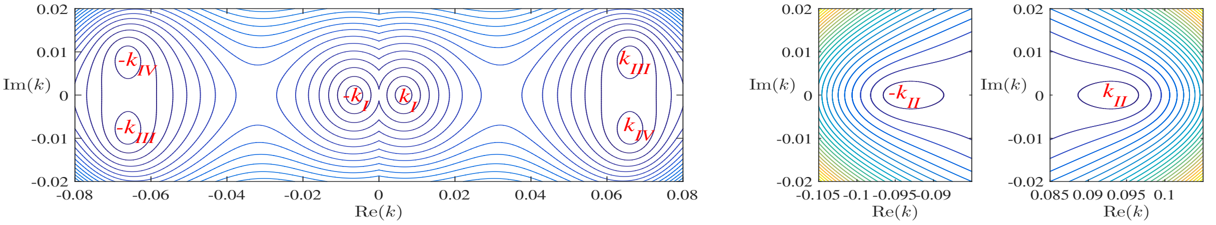
The eigenfunctions $F_n(y)$ are

$$F_n(y) = \begin{cases} g_n(y) = \frac{K\sigma - k_n}{K(\sigma-1)}e^{-k_n(y-h)} + \frac{K-k_n}{K(\sigma-1)}e^{k_n(y-h)} & 0 < y < h \\ e^{k_n(h-y)} & h < y < \infty \end{cases} \quad \text{for } n = I, II, 3, 4.$$

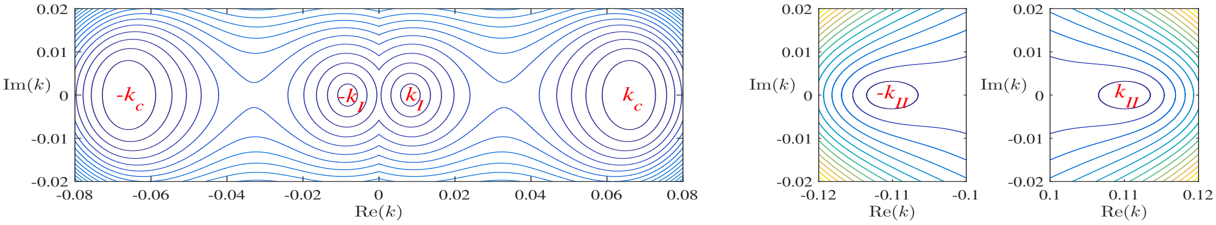
where $\sigma = (1+s)/(1-s)$ and $k_n; (n = I, II, 3, 4)$ represents four propagating modes as discussed in Das et al. [6] , R_n and T_n are the complex constants associated with the amplitude of the reflected and transmitted waves in the n^{th} mode where $\epsilon_n = \pm 1$ when $\frac{d\omega}{dk} \gtrless 0$. Therefore k_n's satisfies the dispersion relation in k as given by

$$(K - k)\left\{k\left(Dk^4 - Qk^2 + 1\right) + K\right\}e^{-kh} + (k - \sigma K)\left\{k\left(Dk^4 - Qk^2 + 1\right) - K\right\}e^{kh} = 0. \tag{8}$$

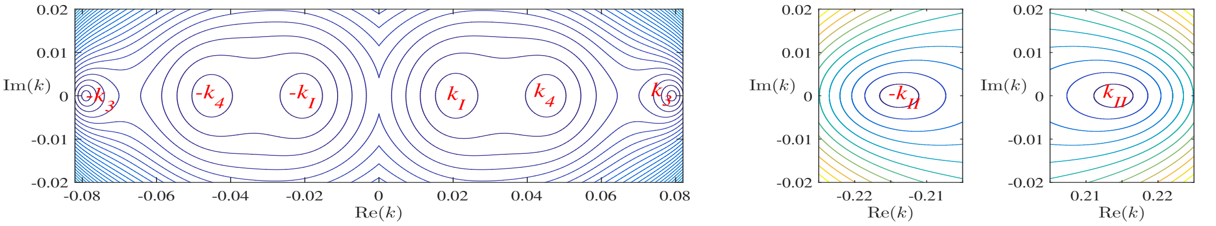
The movement of the roots of dispersion relation in the complex plane with increasing values of incoming wave frequencies is graphically shown in Figure 2. Figure 2(a) reveals that four real roots $\pm k_I$, $\pm k_{II}$ and four complex roots $\pm k_{III}$, $\pm k_{IV}$ with $k_{IV} = \bar{k}_{III}$. As the incoming wave frequency increases, the roots from the complex plane travel towards the real axis and merge to create the secondary blocking (see figure 2(b)). The merging happens for the complex roots k_{III}, $-k_{IV}$ with k_{IV}, $-k_{III}$ respectively, to generate the critical points



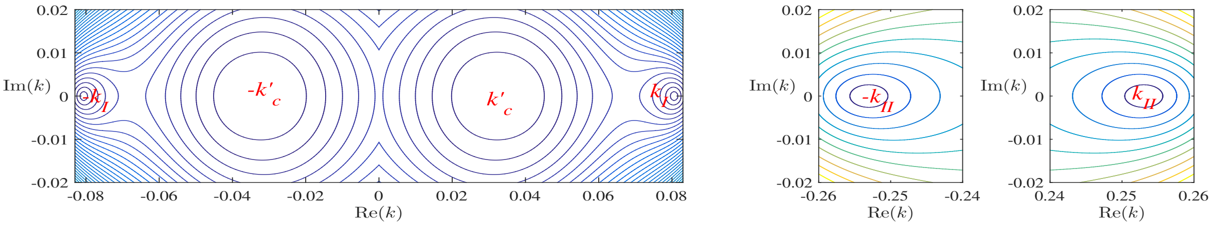
(a) Before secondary blocking



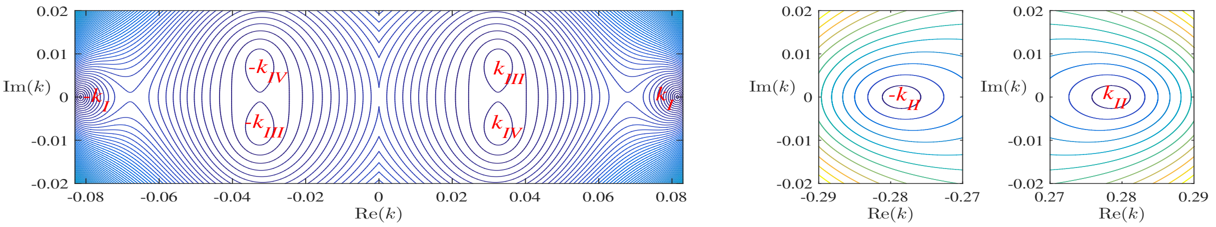
(b) Loci of secondary blocking (modes swapping occur)



(c) Inside the frequency band between primary and secondary blocking.



(d) Loci of primary blocking point (modes swapping occur)



(e) After primary blocking

Figure 2: Contour plot of dispersion relation for different values of incoming wave frequencies are shown. The location of the roots are shown with the small circles which moves in the complex plane with changing frequency (a) for two real roots ($\omega = 0.25\ s^{-1}$), (b) during secondary blocking ($\omega = 0.2808\ s^{-1}$), (c) four real roots ($\omega = 0.41\ s^{-1}$), (d) during primary blocking ($\omega = 0.4475\ s^{-1}$), (e) for two real roots ($\omega = 0.47\ s^{-1}$) when compressive force $Q = 1.95\sqrt{D}$ and density ratio $s = 0.85$ when wave blocking occurs in the surface mode.

$\pm k_c$ that bifurcate into real roots. In order to distinguish the newly generated real roots from the earlier complex ones, we term them as $\pm k_3$ and $\pm k_4$. Then, these roots start travelling away from each other to create in total eight distinct real roots (see figures 2(c)). A further increase in frequency ensures a coalition of the two real roots to generate primary blocking (see figure 2(d)). Coalescence occurs between the roots $\pm k_I$ and $\pm k_4$ to create the critical wave number $\pm k'_c$. Finally, $\pm k'_c$ bifurcates to regenerate the complex roots $\pm k_{III}$ and $\pm k_{IV}$

back into the physical system (see figure 2(e)).
To obtain the energy identity, we use the Green's integral theorem, as given by

$$\int_C \left(\phi^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial \phi^*}{\partial n} \right) ds = 0. \tag{9}$$

where C denotes the closed boundary of the fluid region represented as the union of two closed boundaries C_1 and C_2, ϕ^* is the complex conjugate of ϕ which satisfies Eqs. (1)-(6) and $\frac{\partial}{\partial n}$ represents the outward normal derivative to the closed boundary C. While the closed boundary C_1 consists of the horizontal upper surface $(0 < x < X; y = 0)$, the vertical boundary $(0 < y < \infty; x = X)$, the bottom boundary $(0 < x < X; y \to \infty)$, and the vertical boundary at the crack $(0 < y < \infty; x = 0)$, the closed boundary C_2 consists of the horizontal upper surface $(-X < x < 0;\ y = 0)$, the vertical boundary $(0 < y < \infty;\ x = -X)$, the bottom boundary $(-X < x < 0;\ y \to \infty)$, and the vertical boundary at thecrack $(0 < y < \infty;\ x = 0)$. Ultimately, using the boundary conditions and the far-field radiation condition in (9) and letting $X \to \infty$, the energy relation is obtained as

$$\left(K_{r1}^2 + K_{t1}^2 - 1\right) J_I + \left(K_{r2}^2 + K_{t2}^2 - 1\right) J_{II} = 0, \tag{10}$$

where

$$\begin{aligned}
K_{r1}^2 &= \frac{1}{J_I}\left(|R_I|^2 J_I + |R_3|^2 J_3 - |R_4|^2 J_4\right), \quad K_{t1}^2 = \frac{1}{J_I}\left(|T_I|^2 J_I + |T_3|^2 J_3 - |T_4|^2 J_4\right). \\
K_{r2}^2 &= |R_{II}|^2, \quad K_{t2}^2 = |T_{II}|^2, \\
J_n &= 1 + sk_n \left\{ \frac{P_n^2}{K}\left(4Dk_n^4 - 2Qk_n^2\right) + 2\int_0^h g_n^2(y)dy \right\} \quad \text{for} \quad n = I, II, 3, 4, \\
P_n &= \frac{K\sigma - k_n}{K(\sigma - 1)} e^{k_n h} - \frac{K - k_n}{K(\sigma - 1)} e^{-k_n h} \quad \text{for } n = I, II, 3, 4,
\end{aligned}$$

It may be noted that k_I and k_{II} are incident wave numbers in surface and interface modes respectively. Further, K_{r1}, K_{r2} and K_{t1}, K_{t2} are reflection and transmission coefficients associated with waves in the surface and interface modes respectively. Few results pertaining to the crack problem in the context of energy identity will be discussed during the presentation.

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